Section 8 Markov Chain (from Billingsley's Prob. & measure)

Math 7820
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Transience and persistence.

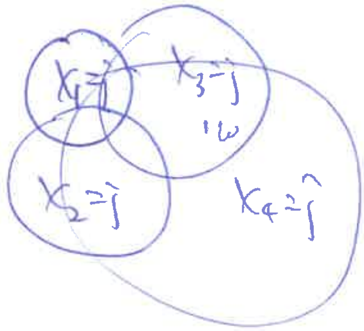
Def: $f_{ij}^{(n)} := \mathbb{P}_i(X_1 \neq j, X_2 \neq j, \cdots X_{n-1} \neq j, X_n = j)$

the prob. of a first visit to j at time n (starting from i).

$$f_{ij} := \mathbb{P}_i\left(\bigcup_{n=1}^{\infty}(X_n = j)\right) = \sum_{n=1}^{+\infty} f_{ij}^{(n)}$$

the prob. of an eventual visit. A state i is persistent if $f_{ii} = 1$. A state i is transient if $f_{ii} < 1$.

Remark: $w \in \bigcup_{n=1}^{+\infty}(X_n = j)$



$\Leftrightarrow \exists\, n \geq 1$ s.t. $X_n(w) = j$.

$$\begin{cases} \bigcup_n \to \exists\, n \\ \bigcap_n \to \forall\, n. \end{cases}$$

Lemma: $\forall i, j$.

$$\mathbb{P}_i(X_n = j \text{ i.o.}) = \begin{cases} 0 & \text{if } f_{jj} < 1 \\ f_{ij} & \text{if } f_{jj} = 1. \end{cases}$$

In particular,

$$\mathbb{P}_i(X_n = i \text{ i.o.}) = \begin{cases} 0 & \text{if } f_{ii} < 1 \text{ (i.e. } i \text{ transient)} \\ f_{ii} & \text{if } f_{ii} = 1. \text{ (i.e. } i \text{ persistent)} \end{cases}$$

Proof: $\mathbb{P}_i(X_n = j \text{ i.o.}) = \mathbb{P}_i\left(\bigcap_{n \geq 1} \bigcup_{m} X_m = j \right)$

Let $1 \leq n_1 < n_2 < \cdots < n_k$ and consider the sytem visits j at $n_1, n_2, \cdots n_k$, but not in between.

$$A \triangleq \{ X_1 \neq j, \cdots X_{n_1 - 1} \neq j, X_{n_1} = j$$
$$X_{n_1+1} \neq j \cdots X_{n_2 - 1} \neq j \quad X_{n_2} = j$$
$$\vdots$$
$$X_{n_{k-1}+1} \neq j \cdots X_{n_k - 1} \neq j \quad X_{n_k} = j \}$$

$$\mathbb{P}(A) = f_{ij}^{(n_1)} f_{jj}^{(n_2 - n_1)} \cdots f_{jj}^{(n_k - n_{k-1})}$$

$\mathbb{P}_i$ (j is visited at least k times)

$$= \sum_{1\le n_1<n_2<\cdots<n_k<+\infty} f_{ij}^{(n_1)}\, f_{jj}^{(n_2-n_1)}\, f_{jj}^{(n_3-n_2)} \cdots f_{jj}^{(n_k-n_{k-1})}$$

$$= \sum_{1\le n_1<n_2<\cdots<n_{k-1}} f_{ij}^{(n_1)}\, f_{jj}^{(n_2-n_1)} \cdots f_{jj}^{(n_{k-1}-n_{k-2})} \underbrace{\sum_{n_k>n_{k-1}} f_{jj}^{(n_k-n_{k-1})}}_{\shortparallel}$$

$$n_k - n_{k-1} = \ell$$

$$\sum_{\ell=1}^{+\infty} f_{jj}^{(\ell)}$$

$$\shortparallel$$

$$f_{jj}$$

$$= \sum_{1\le n_1<n_2<\cdots<n_{k-1}} f_{ij}^{(n_1)}\, f_{jj}^{(n_2-n_1)} \cdots f_{jj}^{(n_{k-1}-n_{k-2})} \cdot f_{jj}$$

$$\vdots$$

$$= f_{ij} \cdot f_{jj}^{k-1}$$

$$P_i(X_n = j \text{ i.o.}) = \begin{cases} 0 & f_{jj} < 1 \\ f_{ij} & f_{jj} = 1 \end{cases}$$

Remark. If i is transient, by lemma above

$$\mathbb{P}_i(X_n = i \text{ i.o.}) = 0 \iff \mathbb{P}_i(X_n \neq i \text{ all but finite}) =$$

i.e. $\exists N$, s.t. $\forall m > N$

$\exists \Omega, P(\Omega=1) \forall w \in \Omega$ $X_m(w) \neq i$.

Thm 8.2.

(i) i is transient $\iff \mathbb{P}_i(X_n = i \text{ i.o.}) = 0$
($f_{ii} < 1$) $\iff \sum_{n=1}^{+\infty} p_{ii}^{(n)} < +\infty$.

(ii) i is persistent ($f_{ii} = 1$) $\iff \mathbb{P}_i(X_n = i \text{ i.o.}) = 1$

$\iff \sum_{n=1}^{+\infty} p_{ii}^{(n)} = +\infty$.

Proof: (i). Assume $\sum_n p_{ii}^{(n)} < +\infty$. $\underset{\text{B.C.I.}}{\Longrightarrow} \mathbb{P}_i(X_n = i, \text{ i.o.}) = 0$

$\underset{\text{lemma above.}}{\Longrightarrow} f_{ii} < 1$ i.e. i is transient.

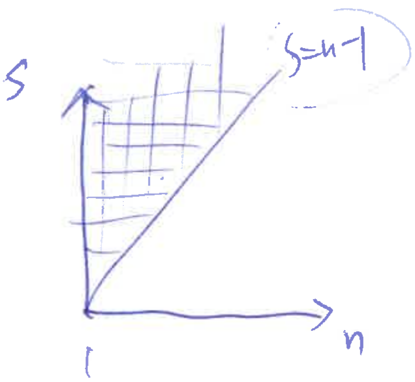
For the other direction, we use first passage argument.

$$p_{ij}^{(n)} = \mathbb{P}_i(X_n = j) = \sum_{s=0}^{n-1} \mathbb{P}_i(X_1 \neq j, \cdots X_{n-s-1} \neq j, \underbrace{X_{n-s} = j}_{s \text{ steps}}, X_n$$

$$= \sum_{s=0}^{n-1} \mathbb{P}_i(X_1 \neq j, \dots, X_{n-s-1} \neq j, X_{n-s} = j)\, \underbrace{\mathbb{P}_j(X_s = j)} \qquad \text{I}$$

$$= \sum_{s=0}^{n-1} f_{ij}^{(n-s)}\, p_{jj}^{(s)}$$

$$\Rightarrow \boxed{\sum_{n=1}^{+\infty} p_{ij}^{(n)}} = \sum_{n=1}^{+\infty} \sum_{s=0}^{n-1} f_{ij}^{(n-s)}\, p_{jj}^{(s)}$$



$$= \sum_{s=0}^{+\infty} p_{jj}^{(s)} \sum_{n=s+1}^{+\infty} f_{ij}^{(n-s)}$$

$n - s = \ell$

$$= \sum_{s=0}^{+\infty} p_{jj}^{(s)} \underbrace{\sum_{\ell=1}^{+\infty} f_{ij}^{(\ell)}}_{f_{ij}}$$

$$= \sum_{s=0}^{+\infty} p_{jj}^{(s)} \cdot f_{ij}$$

$$= f_{ij} + \left(\boxed{\sum_{s=1}^{+\infty} p_{ij}^{(s)}} \right) f_{ij}.$$

$\boxed{\text{Let } j = i}$

$$\Rightarrow (1 - f_{ii}) \sum_{n=1}^{+\infty} p_{ii}^{(n)} = f_{ii}$$

Because $f_{ii} < 1$, ~~the~~ we have

$$1 - f_{ii} > 0$$

$$\Rightarrow \sum_{n=1}^{+\infty} P_{ii}^{(n)} = \frac{f_{ii}}{1-f_{ii}} < +\infty.$$

Part (ii) is just negation of part (i). ☒

<u>Thm 8.3</u> Irreducible Markov chain. Then.

(i) All states are transient.

$$\forall i, \quad \mathbb{P}_i\left(\bigcup_j X_n = j \text{ i.o.} \right) = 0.$$

$$\forall i, j, \quad \sum_n P_{ij}^{(n)} < +\infty$$

(ii) All states are persistent.

$$\forall i \quad \mathbb{P}_i\left(\bigcap_j (X_n = j \text{ i.o.}) \right) = 1$$

$$\forall i, j, \quad \sum_n P_{ij}^{(n)} = +\infty.$$

Remark: Irreducible chain is either a transient chain or a persistent chain.

proof: Since the chain is irreducible, $\forall i, j, \exists r, s$

s.t. $p_{ij}^{(r)} > 0$ and $p_{ji}^{(s)} > 0$.

Now.

$$p_{ii}^{(r+s+n)} \geq \underbrace{p_{ij}^{(r)} \, p_{ji}^{(s)}}_{>0} \, p_{jj}^{(n)}$$

$$\Rightarrow \quad \sum_n p_{ii}^{(r+s+n)} \geq \underbrace{p_{ij}^{(r)} \, p_{ji}^{(s)}}_{>0} \; \underbrace{\sum_n p_{jj}^{(n)}}$$

Hence, $\sum_n p_{ii}^{(r+s+n)} < +\infty \Rightarrow \sum_n p_{ij}^{(n)} < +\infty \overset{\text{Thm 2}}{\Longleftrightarrow} j$ transient ~~per~~

$\Updownarrow$

i is transient

If i is transient,

$\Rightarrow$ all states are transient.

Similarly if i is persistent, then j has to be persistent

i.e. all states are persistent.