Let's continue the proof of Thm 8.3.

(i) If all states are transient, we need to prove

$$\begin{cases} \forall i \quad \mathbb{P}_i\left(\bigcup_j (X_n = j \text{ i.o.})\right) = 0 & \cdots\cdots \text{①} \\ \forall i, j \quad \sum_n P_{ij}^{(n)} < +\infty & \cdots\cdots \text{②} \end{cases}$$

Recall: $\begin{cases} \text{if } \Omega_j \text{ s.t. } P(\Omega_j) = 0 \Rightarrow P\left(\bigcup_j \Omega_j\right) = 0 \\ \text{If } \Omega_j \text{ s.t. } P(\Omega_j) = 1 \Rightarrow P\left(\bigcap \Omega_j\right) = 1. \end{cases}$

(Check Section 2 of Billingsley)

For ①. we only need to show that

$$\mathbb{P}_i(X_n = j, \text{ i.o.}) = 0. \quad \cdots\cdots \text{③}$$

Indeed, we have shown that

$$\mathbb{P}_i(X_n = j. \text{ i.o.}) = \begin{cases} 0 & \text{if } f_{jj} < 1 \\ f_{ij} & \text{if } f_{jj} = 1. \end{cases}$$

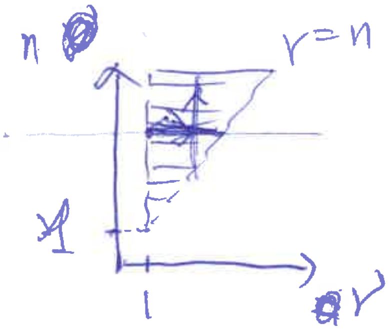
We are in transient case.

Since all states are transient, ③ holds and therefore ① is true.
$(f_{jj} < 1)$

As for ②, $\forall i, j$.

$P, \; P^2, \; P^3, \; P^4 \; \cdots\cdots$ (i,j)-th enty

$$\sum_{n=1}^{+\infty} P_{ij}^{(n)} = \sum_{n=1}^{\infty} \sum_{r=1}^{n} f_{ij}^{(r)} P_{jj}^{(n-r)}$$



$$= \sum_{r=1}^{+\infty} f_{ij}^{(r)} \sum_{n=r}^{+\infty} P_{jj}^{(n-r)}$$

$$= \sum_{r=1}^{+\infty} f_{ij}^{(r)} \left(\sum_{m=0}^{+\infty} P_{jj}^{(m)} \right)$$

$m = n - r$

$m=0$	$n=r$
$m=+\infty$	$n=+\infty$

$$= f_{ij} \cdot \sum_{m=0}^{+\infty} P_{jj}^{(m)}$$

$$\underset{(f_{ij} \le 1)}{\le} \sum_{m=0}^{+\infty} P_{jj}^{(m)} \underset{\text{Thm 8.2. equivalent def. of transient/persistent.}}{<} +\infty$$

~~transient~~

$\sum_{m=0}^{+\infty} P_{jj}^{(m)} = 1 + \underbrace{\sum_{m=1}^{+\infty} P_{jj}^{(m)}}_{<+\infty}$

Therefore, $\sum_{n=1}^{+\infty} P_{ij}^{(n)} < +\infty$, i.e. ② holds.

(ii) Suppose all states are persistent.

By lemma. i.e.

$$P_i(X_n=j \text{ i.o.}) = \begin{cases} 0 & \text{if } f_{jj}<1 \\ f_{ij} & \text{if } f_{jj}=1 \end{cases}$$

$\Leftarrow$ We are in this case. (persistent case)

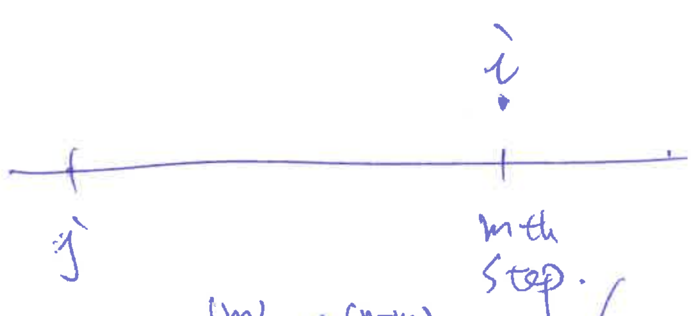
So. $P_i(X_n=j, \text{ i.o.}) = f_{ij}$

Recall Thm 8.2 (equivalent def of persistent): $P_j(X_n=j \text{ i.o.})=1$.

$$p_{ji}^{(m)} = P_j\left((X_m=i)\cap[X_n=j, \text{ i.o.}]\right)$$

$$\| \quad P_j(X_m=i)$$

$$\le \sum_{n>m} P_j(X_m=i, X_{m+1}\neq j, \dots, X_{n-1}\neq j, X_n=j)$$



$$= \sum_{n>m} p_{ji}^{(m)} f_{ij}^{(n-m)} = \Big(\underbrace{\sum_{n>m} f_{ij}^{(n-m)}}_{= f_{ij}}\Big)\cdot p_{ji}^{(m)}$$

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Therefore, we have

$$P_{ji}^{(m)} \le f_{ij} P_{ji}^{(m)} \quad \text{------} \quad (4)$$

~~Recall~~ Because the chain is irreducible, $\exists\, m^{(i,j)}$ s.t.

$$P_{ji}^{(m)} \neq 0.$$

$(4) \Rightarrow \quad f_{ij} \ge 1.$ hence. $f_{ij} \equiv 1.$

By first eq. on P.3, we see that

$$\mathbb{P}_i(X_n = j,\ i.o.) = 1. \quad \forall j \quad \text{---} \quad (5)$$

$$\Rightarrow \mathbb{P}_i\Big(\bigcap_j X_n = j,\ i.o.\Big) = 1.$$

It remains to show that $\sum_n P_{ij}^{(n)} = +\infty.$

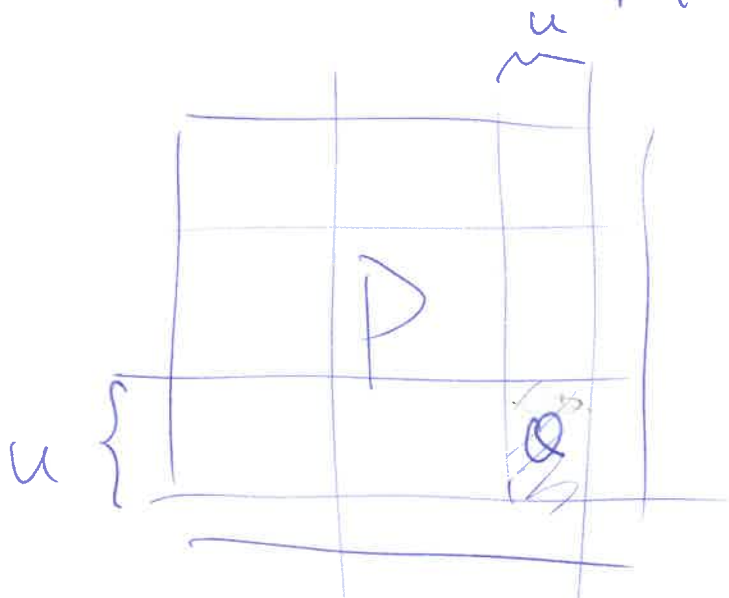
Otherwise, $\sum_n P_{ij}^{(n)} < +\infty$, then by B-C-I, $\mathbb{P}_i(X_n = j, i.o) = 0$

Which contradicts (5) ~~(j = i)~~. ~~(Thm 5.2. equivalent def. i is transient)~~.

Another Criterion for persistence.

P is transition matrix. Let

Q be a submatrix of P. corresponding to states U.



Q is substochastic i.e., $q_{ij} \ge 0$. $\sum_j q_{ij} \overset{\text{Sub.}}{\le} 1$.

$$Q^n = \underbrace{Q \cdots Q}_{n \text{ times}} = [q_{ij}^{(n)}] \qquad Q^0 = [q_{ij}^{(0)}] = I.$$

$$q_{ij}^{(0)} = \delta_{ij} = \begin{cases} 0 & i \ne j \\ 1 & i = j \end{cases}$$

$$\sigma_i^{(n)} := \sum_j q_{ij}^{(n)}$$

Claim 1: $\sigma_i^{(n+1)} = \sum_j q_{ij} \sigma_j^{(n)}$.

Proof: $$q_{ij}^{(n+1)} = \Big(\underbrace{Q^n}_{i\text{th row}} \cdot \underbrace{Q}_{j\text{th column.}}\Big)_{ij}$$

$$= \sum_k q_{ik}^{(n)} q_{kj}$$

Sum over j

$$\sigma_i^{(n+1)} = \sum_k q_{ik}^{(n)} \sum_j q_{kj}$$

$$= \sum_k q_{ik}^{(n)} \sigma_k^{(1)}$$

$$q_{ij}^{(n+1)} = \Big(\underbrace{Q}_{i\text{th row}} \cdot \underbrace{Q^n}_{j\text{th column}}\Big)_{ij}$$

$$= \sum_k \underbrace{q_{ik}}_{i\text{th row}} \underbrace{q_{kj}^{(n)}}_{j\text{th column.}} \qquad n \geq 0$$

We sum over j on both sides.

$$\underbrace{\sum_j q_{ij}^{(n+1)}}_{\sigma_i^{(n+1)}} = \sum_k q_{ik} \underbrace{\sum_j q_{kj}^{(n)}}_{\sigma_k^{(n)}}$$

∎.

Claim 2: $\sigma_i^{(n)}$ is nonincreasing in n.

Proof: $\sigma_i^{(1)} = \sum_j g_{ij} \underset{\substack{\uparrow \\ \text{substochastic}}}{\leq} 1.$

$$\sigma_i^{(n+1)} \underset{\substack{\uparrow \\ \text{eq. on top of p. 6}}}{=} \sum_k g_{ik}^{(n)} \underbrace{\sigma_k^{(1)}}_{\leq 1} \leq \sum_k g_{ik}^{(n)} = \sigma_i^{(n)}$$

$\{\sigma_i^{(n)}, n\}$ is a nonnegative and nonincreasing seq.

$\Rightarrow$ there is a limit:

$$\sigma_i = \lim_{n\to+\infty} \sigma_i^{(n)} = \lim_{n\to+\infty} \sum_j g_{ij}^{(n)}.$$

Claim 3: $\sigma_i = \sum_j g_{ij} \sigma_j$

Proof: From Claim 1, we take limit in n on both sides:

$(\sigma_i^{(n)} \leq 1) \to$ 'DCT' Check.

$$\lim_n \sigma_i^{(n+1)} = \lim_n \sum_j g_{ij} \sigma_j^{(n)}$$

$$\| $$

$$\sigma_i \overset{??}{=} \sum_j \lim_n g_{ij} \sigma_j^{(n)} = \sum_j g_{ij} \sigma_j.$$