Math 7820
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①

Another criterion for persistence.

Q : substochastic ~ $\mathcal{U}$ subset of all states S. $\mathcal{U}\subseteq S$.

$[q_{ij}]$ $\quad q_{ij}\geq 0$ and $\sum_j q_{ij}\leq 1$.

$$\sigma_i^{(n)} = \sum_j q_{ij}^{(n)}$$

Recall: $\sigma_i^{(1)}\leq 1$, $\sigma_i^{(n+1)}\leq \sigma_i^{(n)}$, $\{\sigma_i^{(n)}, n=1,\cdots\}$ is a nonincreasing seq. bounded from below. $\Rightarrow \lim_{n\to\infty}\sigma_i^{(n)} =: \sigma_i$.

Consider the system:

$$\begin{cases} x_i = \sum_{j\in\mathcal{U}} q_{ij}x_j & i\in\mathcal{U} \\ 0\leq x_i\leq 1 & i\in\mathcal{U}. \end{cases} \quad \text{----} (8.24)$$

$(\vec{x}=(x_1,x_2\cdots)\quad \vec{x}\cdot Q = \vec{x})$ ~ Find the ~~right~~ left eigenvector corresponding to eigenvalue 1

<u>Lemma 1</u>: $\{\sigma_i=\lim_n \sigma_i^{(n)}\}$ is the maximal solution to (8.24).

<u>Proof</u>: ① $x_i = \sum_{j\in\mathcal{U}} q_{ij}x_j \underset{\substack{\uparrow\\ 0\leq x_j\leq 1}}{\leq} \sum_{j\in\mathcal{U}} q_{ij} = \sigma_i^{(1)} \quad \forall i\in\mathcal{U}$ ⊛

$$x_i = \sum_{j\in\mathcal{U}} g_{ij}\, x_j \underset{\uparrow \atop \circledast}{\leq} \sum_{j\in\mathcal{U}} g_{ij}\, \sigma_j^{(1)} \underset{\uparrow \atop \text{def. of } \sigma_i^{(n)}}{=} \sigma_i^{(2)}$$

Recall $\sigma_i^{(n+1)} = \sum_{j\in\mathcal{U}} g_{ij}\, \sigma_j^{(n)}$

$\vdots$

$x_i \leq \sigma_i^{(n)}$ for all $n \geq 1$.

Therefore, $x_i \leq \inf_n \sigma_i^{(n)} = \lim_n \sigma_i^{(n)} = \sigma_i \quad \forall i \in \mathcal{U}$.

② Notice that

$$\begin{array}{ccc} \sigma_i^{(n+1)} & = & \sum_{j\in\mathcal{U}} g_{ij}\, \sigma_j^{(n)} \\ \downarrow & & \downarrow \\ \sigma_i & & \sum_{j\in\mathcal{U}} g_{ij}\, \sigma_j \end{array}$$

M-test (Dominated Convergence).

$n\to+\infty$

i.e.,
$$\begin{cases} \sigma_i = \sum_{j\in\mathcal{U}} g_{ij}\, \sigma_j & i\in\mathcal{U} \\ 0 \leq \sigma_i \leq 1 & i\in\mathcal{U}. \end{cases}$$

So. $\{\sigma_i\}$ is a solution. to (8.24).

By ①&②, Therefore, $\{\sigma_i\}$ is the maximal sol to (8.24). 証

$$\sigma_i^{(n)} = \sum_{\substack{j_1 \cdots j_n \\ \in \\ \mathcal{U} \\ \subseteq \\ S}} P_{ij_1} P_{j_1 j_2} P_{j_2 j_3} \cdots P_{j_{n-1} j_n}$$

$$= \mathbb{P}_i\{X_t \in \mathcal{U},\ t \le n\}.$$

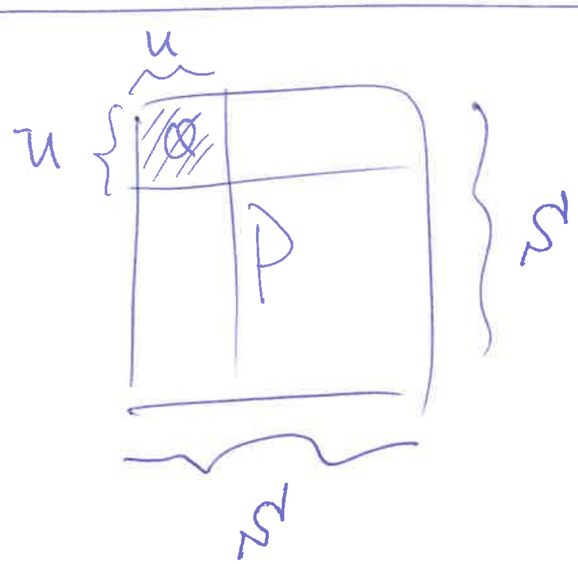
$$\sigma_i = \lim_{n \to +\infty} \sigma_i^{(n)} = \mathbb{P}_i\{X_t \in \mathcal{U},\ t = 1, 2, 3, \dots\}.\quad i \in \mathcal{U}.$$

↑ prob. of staying in $\mathcal{U}$ forever.

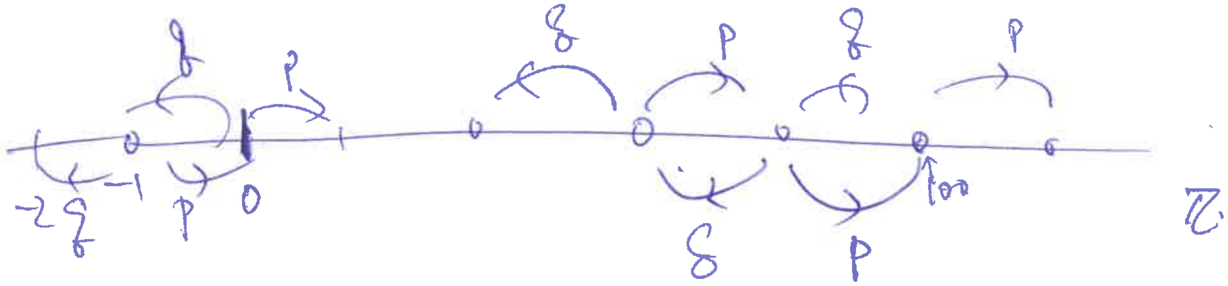
Lemma 1 implies

<u>Thm 8.4</u>: For $\mathcal{U} \subseteq S$, the prob. σ_i are maximal sol. to the system.

$$\begin{cases} x_i = \sum_{j \in \mathcal{U}} P_{ij} x_j & i \in \mathcal{U} \\ 0 \le x_i \le 1 & i \in \mathcal{U}. \end{cases}$$



Example 8.10. Random walk on the line.



$$\begin{cases} x_i = p\,x_{i+1} + q\,x_{i-1} & i \geq 1 \\ x_0 = p\,x_1 & i = 0 \end{cases}$$

$(p+q=1)$

$U = \mathbb{N}_0 = \{0, 1, 2 \cdots\}$

$\subseteq S = \mathbb{Z}$

$$\begin{array}{c|ccccc} & 0 & 1 & 2 & 3 & 4 \cdots \\ \hline 0 & 0 & p & & & \\ 1 & q & 0 & p & & \\ 2 & & q & 0 & p & \\ 3 & & & q & 0 & p \\ 4 & & & \cancel{q} & q & 0 \\ \vdots & & & & & \end{array}$$

If $p = q$, $x_n = A + An.$

$\shortparallel$

$\frac{1}{2}$

$\text{Left side} = A + Ai \overset{??}{=} p\,(A + A(i+1)) + p\,(A + A(i-1))$

Right Side

$\shortparallel$

$2pA + 2pAi$

$\shortparallel \quad p = \frac{1}{2}$

$A + Ai$

(Check [A19] for the way to solve)

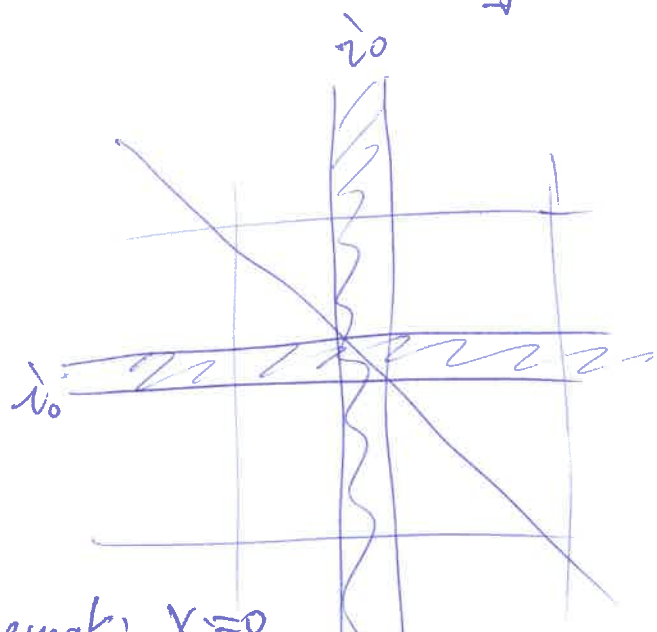
If $p \neq q$, $X_n = A - A\left(\frac{q}{p}\right)^{n+1}$. (Chapter 0 of Lawler's book).

If $q \geq p$, the only bounded sol. is $x_n = 0$.

i.e., with probability zero, particle will stay forever in nonnegative integers.

If $q < p$, $X_n = 1 - \left(\frac{q}{p}\right)^{n+1}$

图



$U = S \setminus \{i_0\}$.

$$(8.27)\quad \begin{cases} x_i = \sum_{j \neq i_0} P_{ij} x_j & i \neq i_0 \\ 0 \leq x_i \leq 1 & i \neq i_0 \end{cases}$$

Remark: $x_i = 0$ is always a trivial sol.

<u>Thm 8.5</u> An irreducible chain is transient iff (8.27) has a nontrivial sol.

(In other words, persistent iff ~~(8.27)~~ the only sol. to (8.27) is the trivial sol.)

<u>Proof</u>: "$\Leftarrow$" Suppose (8.27) has a nontrivial sol.

$$1 - f_{i i_0} = P_i(X_n \neq i_0,\ n \geq 1) = \sigma_i$$

Should be the maximal sol. to (8.27) (by Thm 8.4). ⑥

(8.27) has a nontrivial sol. iff $f_{ii_0} < 1$ for some $i \neq i_0$

$\Updownarrow$

$v_i \neq 0$ for some $i \neq i_0$

$\Updownarrow$

$f_{ii_0} < 1$ for some $i \neq i_0$

$\Rightarrow$ the ~~chain~~ is transient.

~~because otherwise,~~ if the chain is persistent, $\Rightarrow f_{ii_0} = 1$.
By Thm 8.3(ii)

A contradiction.

"$\Rightarrow$" Suppose the chain is transient.

$$f_{i_0 i_0} = \mathbb{P}_{i_0}(X_1 = i_0) + \sum_{n=2}^{+\infty} \sum_{i \neq i_0} \mathbb{P}_{i_0}(X_1 = i, X_2 \neq i_0 \cdots X_{n-1} \neq i_0, X_n = i_0)$$

$$= P_{i_0 i_0} + \sum_{i \neq i_0} P_{i_0 i} \underbrace{f_{ii_0}}_{<1} < P_{i_0 i_0} + \sum_{i \neq i_0} P_{i_0 i} = 1.$$

(transient def. of)

Since $f_{i_0 i_0} < 1$, it follows that $f_{ii_0} < 1$ for some $i \neq i_0$ ∎