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Math 7820
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Example 8.11 Queueing Models.

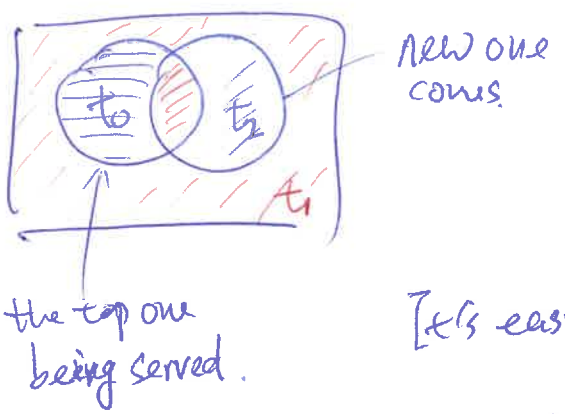
$S = \{0, 1, 2, \cdots\}$

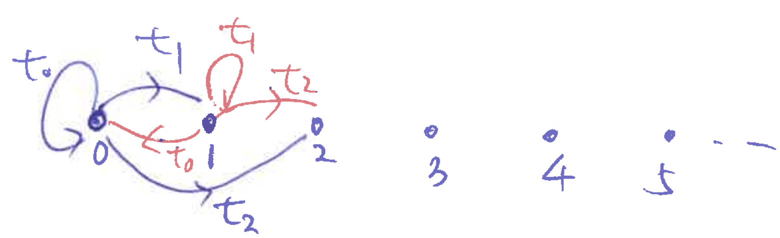
↑ # of customs

Stochastic Matrix.

$$\begin{array}{c|c} & \begin{array}{cccccc} 0 & 1 & 2 & 3 & 4 & \end{array} \\ \hline \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ \vdots \end{array} & \left[\begin{array}{cccccc} t_0 & t_1 & t_2 & 0 & 0 & 0 \cdots \\ t_0 & t_1 & t_2 & 0 & 0 & 0 \cdots \\ 0 & t_0 & t_1 & t_2 & 0 & 0 \cdots \\ 0 & 0 & t_0 & t_1 & t_2 & 0 \cdots \\ 0 & 0 & 0 & t_0 & t_1 & t_2 \cdots \\ \cdots & & & & & \end{array}\right] \end{array}$$

← Q. Substochastic.





It's easy to see that the chain is irreducible.

Now we remove $i_0 = 0$ and apply thm 8.5 by studying

$$\begin{cases} x_1 = t_1 x_1 + t_2 x_2 \\ x_k = t_0 x_{k-1} + t_1 x_k + t_2 x_{k+1} \qquad k = 2, 3 \cdots \\ x_i \in [0,1] \quad i = 0, 1, 2 \cdots \end{cases}$$

By Chapter 0 of Lawler's book, [A19].

$$\begin{cases} (1-t_1)\, x_1 = t_2\, x_2 \\ (1-t_1)\, x_k = t_0\, x_{k-1} + t_2\, x_{k+1} \end{cases}$$

$$\Longleftrightarrow \begin{cases} x_1 = \dfrac{t_2}{1-t_1}\, x_2 & \text{---- eq.(1)} \\ x_k = \underbrace{\dfrac{t_0}{1-t_1}}_{q}\, x_{k-1} + \underbrace{\dfrac{t_2}{1-t_1}}_{p}\, x_{k+1} & \text{---- eq.(2)} \quad k=2,3,\dots \end{cases}$$

Recall ~~By~~ [A19]

$$x_n = p\, x_{n+1} + q\, x_{n-1} \qquad a<n<b \qquad (p+q=1)$$

$$\Longrightarrow \quad x_n = \begin{cases} A + B\left(\frac{q}{p}\right)^n & a \le n \le b \qquad p \ne q \\ A + Bn & a \le n \le b \qquad p = q = \frac{1}{2} \end{cases}$$

$$\cancel{\alpha^2 - \tfrac{1}{2}\alpha - \tfrac{1}{2} = 0} \qquad \cancel{\left(\alpha - \tfrac{1}{2}\right)^2 + \tfrac{3}{4}}$$

$$\frac{1}{2}\alpha^2 - \alpha + \frac{1}{2} = 0 \iff (\alpha-1)^2 = 0, \quad \alpha_1 = \alpha_2 = 1.$$

Check Lawler's Chapter 0.

By eq. ② on p. 2. we have.

$$X_k = \begin{cases} A + B\left(\frac{t_0}{t_2}\right)^n & t_0 \neq t_2. \\ A + Bk & t_0 = t_2. \end{cases} \qquad k = 2, 3, \dots$$

We still need to use eq. ① on p. 2. which will determine A:

If $t_0 \neq t_2$.
$$\underbrace{A + B\left(\frac{t_0}{t_2}\right)^1}_{= X_1} = \frac{t_2}{1-t_1}\left(\underbrace{A + B\left(\frac{t_0}{t_2}\right)^2}_{= X_2}\right)$$

$$\left(1 - \frac{t_2}{1-t_1}\right)A = B\left[\left(\frac{t_0}{t_2}\right) + \frac{t_2}{1-t_1}\left(\frac{t_0}{t_2}\right)^2\right]$$

$$\frac{\overset{t_0}{\overset{\parallel}{1-t_1-t_2}}}{1-t_1} A = B\left(\frac{t_0}{\cancel{t_2}}\right)\underbrace{\left[-1 + \frac{\cancel{t_2}}{1-t_1}\frac{t_0}{\cancel{t_2}}\right]}_{\frac{-1+t_1+t_0}{1-t_1} = -\cancel{t_2}}$$

$$\frac{t_0}{1-t_1} A = \frac{-t_0}{1-t_1} B$$

$$A = -B.$$

$$\frac{t_2}{1-t_1} = \frac{t_2}{t_0+t_2} = \frac{1}{2}$$

If $t_0 = t_2$
$$\underbrace{A + B \times 1}_{x_1} = \frac{t_2}{1-t_1}\left(\underbrace{A + B \times 2}_{x_2}\right)$$

$$A + B = \frac{1}{2}(A + 2B)$$
$$\frac{1}{2}A = 0$$
$$A = 0$$

Therefore. If $t_0 \neq t_2$.

$$X_k = \cancel{Q}\, B\left(\left(\frac{t_0}{t_2}\right)^n - 1\right) \qquad k = 2, 3 \ldots$$

If $t_0 = t_2$.

$$X_k = Bk. \qquad k = 2.3 \ldots$$

Therefore, $\exists$ a nontrivial sol if $t_0 < t_2$ $\Longleftrightarrow$ transient.

but NOT if $t_0 \geq t_2$. $\Longleftrightarrow$ persistent

$\blacksquare$

Stationary Distributions:

Def: ~~Let~~ If the initial probabilities $\{\pi_i\}$ is such that

$$\sum_{i \in S} \pi_i P_{ij} = \pi_j \qquad j \in S.$$

then $\{\pi_i\}$ is called the stationary distr. of the chain.

$$\vec{\pi} \; \boxed{P} = \vec{\pi}$$

$\vec{\pi}$ is the left eigenvector of P corresponding to $\lambda = 1$.
(Right eigenvector $(\vdots)$).

Why stationary?

$$\vec{\pi} P^n = \underbrace{\vec{\pi} P}\cdot P^{n-1}$$
$$= \vec{\pi} P^{n-1}$$
$$= \vec{\pi} P^{n-2}$$
$$\vdots$$
$$= \vec{\pi}.$$

Def: The **period** of $j \in S$ is the greatest common divisor of the set of integers:

$$\{n : n \geq 1 \ \ P_{jj}^{(n)} > 0\}.$$

Claim: If the chain is irreducible, all states have the same period.

(See Notes from Lawler's book).

Def: An irreducible chain is called aperiodic if its period is one.

Lemma 2. In an irreducible ~~chain~~, aperiodic chain, ⑥

for $\forall i, j \in S$, $\underbrace{\quad}_{\exists\, n_0(i,j) \text{ s.t.}}$ $P_{ij}^{(n)} > 0$ for all $n \geq n_0(i,j)$.

Proof: Since $P_{jj}^{(m+n)} \geq P_{jj}^{(m)} P_{jj}^{(n)}$, if

$$M = \{ n : n \geq 1,\ P_{jj}^{(n)} > 0 \}.$$

and $m \in M$, $n \in M$, $\Rightarrow$ $m+n \in M$.

Therefore, M is closed under addition.

~~A fact from Number theory,~~ $\Rightarrow$ ~~greatest common divisor of M is one.~~ $\Rightarrow$

$+$ M has greatest common divisor being one.

$\uparrow$ Number theory. [A21]

$\exists\, n_1$ s.t. $\forall\, n \geq n_1$, $n \in M$.

$\forall\, i, j \in S$. $\exists\, r$, $P_{ij}^{(r)} > 0$. Now. if $n > n_0 = n_1 + r$

$\uparrow$ irreducible.

$n-r \in M$

then. $P_{ij}^{(n)} \geq \underbrace{P_{ij}^{(r)}}_{>0} \underbrace{P_{jj}^{(n-r)}}_{>0.} > 0.$ ⓠ.