Stationary distribution

Final Aim is Thm 8.8

Recall $\boxed{\text{Stationary distr.}}$

$$\sum_{i\in S} \pi_i P_{ij} = \pi_j$$

$\forall j \in S$. where

$\pi_i \geq 0$ s.t. $\sum_{i\in S} \pi_i = 1$.

Irreducible, and aperiodic chain

$$\begin{cases} \text{transient} \\ \text{persistent.} \begin{cases} \text{null persistent} \\ \text{positive persistent. } (\exists \text{ Stationary distr.}) \end{cases} \end{cases}$$

Thm 8.6 Suppose the ~~chain~~ chain is irreducible and aperiodic.

If there exists a stationary distribution, then

$$\begin{cases} \text{(i) the chain is persistent.} \\ \text{(ii) } \pi_j = \lim_{n\to\infty} P_{ij}^{(n)}. \quad \forall i,j \in S \qquad P^{(n)} \text{ n large} = \begin{pmatrix} \pi_1 & \cdots & \pi_{\cdot} \\ & \vdots & \end{pmatrix} \\ \text{(iii) the stationary distr. is unique.} \end{cases}$$

Thm 8.7 Suppose the ~~chain~~ chain is irreducible & aperiodic.

If there is NO stationary distr., then

(I) $\lim_{n\to+\infty} P_{ij}^{(n)} = 0. \quad \forall i,j \in S.$

proof of thm 8.6.

(i) If the chain is transient, then by Thm 8.3, $\forall\, i, j \in \mathcal{S}$.

b.c. the chain is irreducible

$P_{ij}^{(n)} \to 0$ as $n \to +\infty$. Since the chain has a stationary dist

then

$$\sum_{i \in \mathcal{S}} \pi_i P_{ij}^{(n)} = \pi_j \qquad \forall\, i, j \in \mathcal{S}$$

Take the limit $n \to +\infty$ on both side, by dominated convergence / M-test,

$$\sum_{i \in \mathcal{S}} \pi_i \underbrace{\lim_{n \to \infty} P_{ij}^{(n)}}_{\substack{\parallel \\ 0}} = \pi_j$$

$\Rightarrow \pi_j \equiv 0. \ \forall j \in \mathcal{S}$. which contradicts with $\sum_j \pi_j = 1$

Therefore, the chain has to be persistent.

(ii), (iii): Consider a chain on $\mathcal{S} \times \mathcal{S}$. with transition prob:

$$p(ij, kl) = P_{ik} P_{jl}. \qquad \sim \text{ coupled chain.}$$

(X_n, Y_n), $n = 0, \cdots$

$$\mathbb{P}\big((X_{n+1}, Y_{n+1}) = (k, l) \mid (X_n, Y_n) = (i, j) \big) = p(ij, kl).$$

B.c. the original chain is irreducible, the same is true for the coupled chain, that is, $\forall (k,l),(i,j)\ \exists\, n_0$ s.t.

$$\forall n \geq n_0 \quad p^{(n)}(ij, kl) > 0.$$

$$\parallel$$

$$p_{ik}^{(n)}\, p_{jl}^{(n)}$$

$\exists\, n_1(i,k)$ s.t. $\forall n \geq n_1\ p_{ik}^{(n)} > 0$

$\exists\, n_2(j,l)$ s.t. $\forall n \geq n_2\ p_{jl}^{(n)} > 0$

irreducible + aperiodic (Lemma 2)

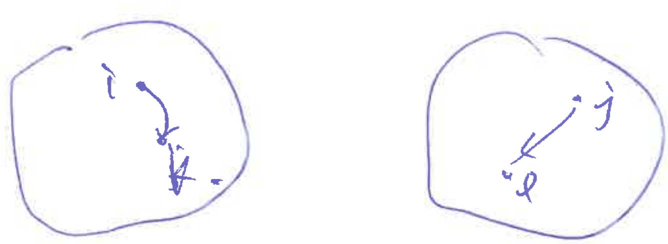
$\Rightarrow$ Let $n_0 = n_1 \vee n_2$. then $\forall n > n_0$

$p_{ik}^{(n)} > 0$ and $p_{jl}^{(n)} > 0$.

$\Rightarrow p^{(n)}(ij, kl) > 0$.

Therefore, the coupled chain is irreducible and aperiodic.

Because "$Y_n \perp X_n$", we see that $\pi(ij) = \pi_i \pi_j$ is the stationary distr. of the coupled chain.



X_n $\qquad\qquad$ Y_n

Therefore, by part (i), we conclude that the coupled chain is persistent.

(4)

B.C. the couple chain is persistent.

$\forall i,j \in S\times S$, $\forall i_0 \in S$

$$\mathbb{P}_{ij}\big((X_n, Y_n) = (i_0, i_0) \ \text{i.o.} \big) = 1. \quad (\text{Thm 8.3})$$

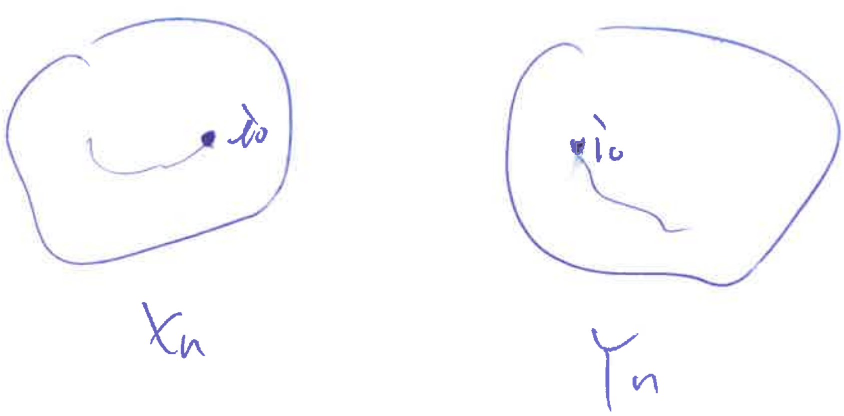
Let τ be the first time (smallest integer) s.t.

$$X_\tau = Y_\tau = i_0.$$

[i.e., $\tau = \min\{n : X_n = Y_n = i_0\}$.]

<u>Claim</u>: $\tau < +\infty$ a.s.

After τ:



If $m \le n$.

$$P_{ij}\big((X_n, Y_n) = (k,l),\ \tau = m \big)$$

$$= P_{ij}\big((X_t, Y_t) \neq (i_0, i_0)\ t<m,\ (X_m, Y_m) = (i_0, i_0) \big)$$

$$\times P_{i_0 i_0}\big((X_{n-m}, Y_{n-m}) = (k,l) \big).$$

$$= \mathbb{P}_{ij}(\tau = m)\ P^{(n-m)}_{i_0 k}\ P^{(n-m)}_{i_0 \ell}$$

$\uparrow$ X chain $\perp$ $\uparrow$ Y chain

Summation over ℓ on ~~both~~ sides, gives

$$P_{ij}(X_n = k, \tau = m) = P_{ij}(\tau = m)\, P^{(n-m)}_{i_0 k}.$$

Similarly, Summation over k on both sides gives

$$P_{ij}(Y_n = \ell, \tau = m) = P_{ij}(\tau = m)\, P^{(n-m)}_{i_0 \ell}$$

Now take $k = \ell$.

$$P_{ij}(X_n = k, \tau = m) = P_{ij}(Y_n = \ell, \tau = m).$$

$\Rightarrow$

$$P_{ij}(X_n = k, \tau \le n) = P_{ij}(Y_n = k, \tau \le n).$$

Therefore :

$$\begin{aligned} P_{ij}(X_n = k) &\le P_{ij}(X_n = k, \tau \le n) + \mathbb{P}_{ij}(\tau > n). \\ &= P_{ij}(Y_n = k, \tau \le n) + P_{ij}(\tau > n) \\ &\le P_{ij}(Y_n = k) + P_{ij}(\tau > n) \end{aligned}$$

Therefore,

$$P_{ij}(X_n=k) \le P_{ij}(Y_n=\cancel{\ell}k) + P_{ij}(\tau>n)$$

By symmetry,

$$P_{ij}(Y_n=\cancel{\ell}k) \le P_{ij}(X_n=k) + P_{ij}(\tau>n).$$

Therefore,

$$\left|\underbrace{P_{ij}(X_n=k) - P_{ij}(Y_n=\cancel{\ell}k)}_{\begin{gathered}\shortparallel \\ \left|P_{ik}^{(n)} - P_{j\cancel{\ell}k}^{(n)}\right|\end{gathered}}\right| \le P_{ij}(\tau>n).$$

Because $\tau<+\infty$ a.s., by sending $n\to\infty$ on both sides,

$$\lim_{n\to\infty}\left|P_{ik}^{(n)} - P_{j\cancel{\ell}k}^{(n)}\right| \le \lim_{n\to\infty} P_{ij}(\tau>n) = 0.$$

Recall

$$\sum_{i\in S} \pi_i P_{ik}^{(n)} = \pi_k$$

$$\pi_k - P_{jk}^{(n)} = \sum_{i\in S} \pi_i \left(P_{ik}^{(n)} - P_{jk}^{(n)} \right).$$

Send $n \to +\infty$ on both sides and use DCT/M-test,

$$\lim_{n\to+\infty} \left(\pi_k - P_{jk}^{(n)}\right) \underset{\text{DCT/M-test}}{\overset{\uparrow}{=}} \sum_{j\in S} \pi_i \underbrace{\lim_{n\to\infty} \left(P_{ik}^{(n)} - P_{jk}^{(n)}\right)}_{\substack{\text{|||} \\ 0}} = 0$$

Therefore, $\lim_{n\to\infty} P_{jk}^{(n)} = \pi_k. \quad \forall j,k \in S.$

By construction, this π distr. is unique,

It remains to show that π_j are indeed stationary distr.

by showing that all π_j are strictly positive. $\pi_j \geq 0$.

Recall: $\forall i \in S$

$$P_{ii}^{(r+s+n)} \geq \underset{\underset{0}{\vee}}{P_{ij}^{(r)}} P_{jj}^{(n)} \underset{\underset{0}{\vee}}{P_{ji}^{(s)}} \qquad . \quad \pi_i > P_{ij}^{(r)} \cdot P_{ji}^{(s)} \cdot \pi_j > 0$$

by choosing r, s large, $\overset{\text{irreduc.}}{\Longrightarrow}$ then send $n \to +\infty$. ☺