Math 7820 Auburn
2025/09/25 Le Chen



**Thm 8.7.** Suppose the chain is irreducible & aperiodic.

If there is no stationary distr; then

$$\lim_{n\to\infty} P_{ij}^{(n)} = 0 \qquad \forall\, i,j. \qquad \begin{cases} \text{transient} \\ \text{null-persistent} \end{cases}$$

~~Proof~~ Remark: If the chain is transient, then by thm 8.3 $\lim_{n\to\infty} P_{ij}^{(n)} = 0$. Therefore, the theorem is interesting only in case ~~there the~~ of persistent chain.

Proof: By Remark above, we'll assume that the chain is persistent. Consider the coupled chain built in the proof of thm 8.6, which is irreducible, aperiodic and persistent. If $P_{ij}^{(n)}$ do not go to zero as $n\to+\infty$, then $\exists$ an increasing sequence $\{n_u\}$ of integers along which $P_{ij}^{(n_u)}$ is bounded away from zero. By diagonal Argument, it is possible ~~by~~ passing to a subsequence of $\{n_u\}$ s.t. there is a limit. For convenience, we use $\{n_u\}$ for this further subsequence. Then $\lim_{u\to\infty} P_{ij}^{(n_u)} = t_j$, $\forall\, i,j \in S$.

$t_j \geq 0.\ \forall j \in S.$ and for some $j \in S$, $t_j > 0.$

because the chain is persistent.

Let $M \subseteq S$.

$$\sum_{j \in M} t_j = \lim_{u \to +\infty} \sum_{j \in M} p_{ij}^{(n_u)} \leq 1.$$

Switch ✓
DCT/M-test

$\Downarrow$

$$\boxed{0 < t := \sum_{j \in S} t_j \leq 1 \implies \sum_{j \in S} \frac{t_j}{t} = 1.}$$

Now.

$$\sum_{k \in M} p_{ik}^{(n_u)} p_{kj} \underset{\uparrow\, M \subseteq S}{\leq} p_{ij}^{(n_u+1)} = \sum_{k \in S} p_{ik} p_{kj}^{(n_u)}$$

Take lim $u \to +\infty$ on both sides (Switch $\Sigma$ and lim by M-test/DCT)

$$\sum_{k \in M} t_k p_{kj} \leq \overbrace{\sum_{k \in S} p_{ik}}^{=1} t_j = t_j \qquad \forall M \subseteq S.$$

In particular, by choosing $M = S$,

$$\sum_{k \in S} t_k p_{kj} \leq t_j$$

The above inequality is indeed an equality.

Otherwise. If $\sum_{k\in S} t_k P_{kj} < t_j$, sum over $j \in S$ on both sides gives:

$$\underbrace{\sum_{k,j\in S} t_k P_{kj}}_{=\sum_{k\in S} t_k} < \sum_{j\in S} t_j$$

which is not possible.

Therefore,

$$\sum_{k\in S} \underbrace{\frac{t_k}{t}}_{\pi_k} P_{kj} = \underbrace{\frac{t_j}{t}}_{\pi_j}. \quad \forall j \in S.$$

$$\sum_{j\in S} \underbrace{\frac{t_j}{t}}_{\pi_j} = 1$$

Therefore, we find a stationary distribution of the chain: $\pi_j = \frac{t_j}{t}$, $j \in S$.

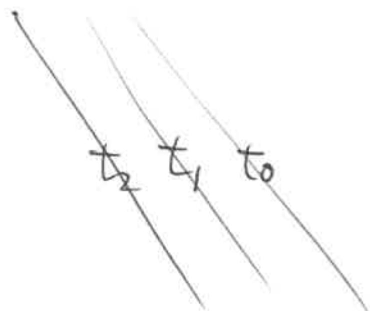
This contradicts with no stationary distr. assumption.

Example 8.12. Queueing model.

$$\begin{cases} \pi_0 = \pi_0 t_0 + \pi_1 t_0 \\ \pi_1 = \pi_0 t_1 + \pi_1 t_1 + \pi_2 t_0 \\ \pi_2 = \pi_0 t_2 + \pi_1 t_2 + \pi_2 t_1 + \pi_3 t_0 \\ \boxed{\pi_k = \pi_{k-1} t_2 + \pi_k t_1 + \pi_{k+1} t_0 \qquad k \geq 3} \end{cases}$$

bdry.

(Typos in the book??)




$$\pi_k = \begin{cases} A + B\left(\frac{t_2}{t_0}\right)^k & \text{if } t_0 \neq t_2 \\ A + Bk. & \text{if } t_0 = t_2. \end{cases}$$

If $t_0 < t_2$, and $\sum_{k=0}^{+\infty} \pi_k$ converges $\Longrightarrow$ $\pi_k \equiv 0$ i.e. $A = B = 0$.

i.e. there is No stationary distr.

$\uparrow$

trivial sol is the only sol.

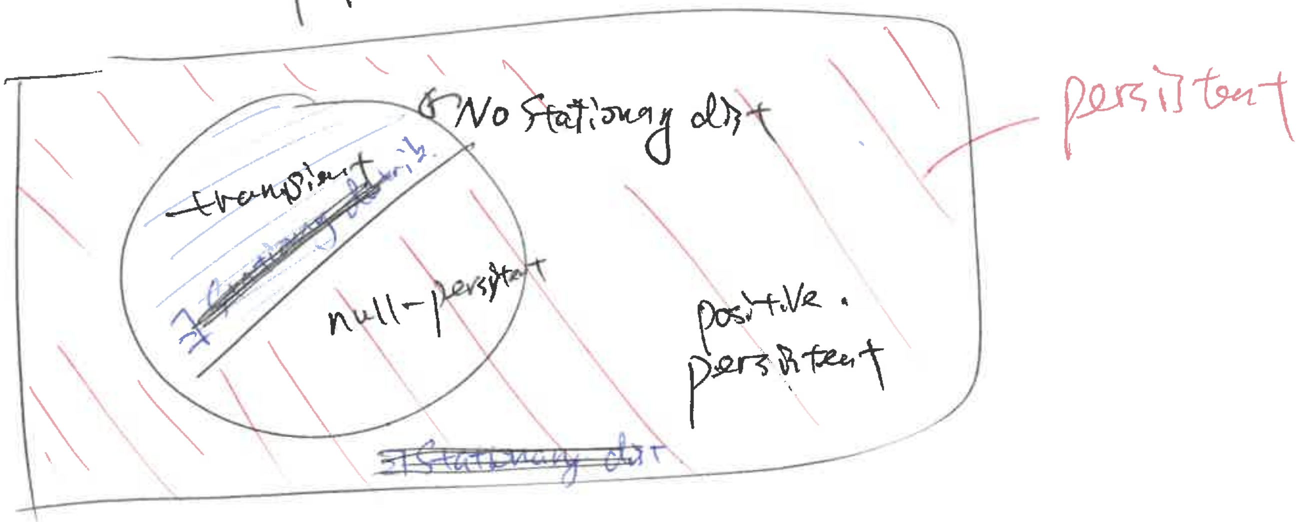
$\Updownarrow$ Thm 8.8

the chain is ~~transient~~ persistent.

If $t_0 = t_2$, by Example 8.11, the chain is still persistent. but in this case $A = B = 0$. that is trivial sol. is the only sol.

So if $t_0 = t_2$, there is No stationary distr.

This kind of persistent is called null-persistent.




If $t_0 > t_2$, we can find a nontrivial sol. $A=0$. then

$$\sum_{k=0}^{+\infty}\pi_k = \sum_{k=0}^{+\infty} B\left(\frac{t_2}{t_0}\right)^k = B\cdot\frac{1}{1-\frac{t_2}{t_0}} = \frac{Bt_0}{t_0-t_2} < +\infty$$

$$\frac{Bt_0}{t_0-t_2} = 1$$

$$\Rightarrow B = \frac{t_0-t_2}{t_0}.$$

So. stationary distr.

$$\pi_k = \frac{t_0-t_2}{t_0}\cdot\left(\frac{t_2}{t_0}\right)^k,$$