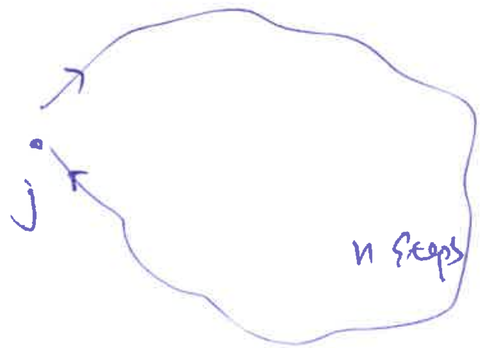
Mean return times.

Recall $f_{jj}^{(n)}$: is the prob. of starting from j and returning to j for the first time in n steps.



n	$n=1$	$n=2$	...
$f_{jj}^{(n)}$			

Mean return time $\mu_j := \sum_{n=1}^{+\infty} n f_{jj}^{(n)}$

It may happen that $\mu_j = \infty$.

Lemma 3 : Suppose that j is persistent and $\lim_{n\to\infty} P_{jj}^{(n)} = u$.

Then $u > 0 \iff \mu_j < +\infty$ $\left(u = \frac{1}{\mu_j} \right)$.

Proof: For $k \geq 0$, and j fixed, let $\rho_k := \sum_{n>k} f_{jj}^{(n)}$.

Consider the double series.

$$\begin{array}{ccccccccc} & & & & & & \xrightarrow{\quad\quad\quad} n & & \\ f_{jj}^{(1)} & + & f_{jj}^{(2)} & + & f_{jj}^{(3)} & + & \cdots & + & \ell_0 \\ & & & & & & & & + \\ & + & f_{jj}^{(2)} & + & f_{jj}^{(3)} & + & \cdots & + & \ell_1 \\ & & & & & & & & + \\ k \downarrow & & & + & f_{jj}^{(3)} & + & \cdots & + & \ell_2 \\ \hline \uparrow & & \uparrow & & \uparrow & & & & \parallel \end{array}$$

$$\mu_j = 1 \times f_{jj}^{(1)} + 2 \times f_{jj}^{(2)} + 3 \times f_{jj}^{(3)} + \cdots$$

$$\boxed{\mu_j = \sum_{k=0}^{+\infty} \ell_k}$$

Since j is persistent, the P_j-probability that ~~that system~~ does not hit j up to time n is the prob. that it hits j after time n. and this is ℓ_n.

$$\sum_{k>n} f_{jj}^{(k)} = \ell_n.$$

Therefore,

$$1 - \underbrace{p_{jj}^{(n)}}_{\ell_0 \times,\ \ell_0 = 1} = P_j(X_n \neq j).$$

$$= P_j(X_1 \neq j, X_2 \neq j, \cdots X_n \neq j)$$

$$+ \sum_{k=1}^{n-1} P_j(\underset{\uparrow}{X_k} = j, X_{k+1} \neq j, \cdots, X_n \neq j)$$

the last visit of j before n steps.

$$= \overbrace{\ell_n}^{\times p_{jj}^{(0)} = 1} + \sum_{k=1}^{n-1} p_{jj}^{(k)} \ell_{n-k}$$

Note: $\ell_0 = 1$. $\left(\ell_0 = \sum_{n>0} f_{jj}^{(n)}\right)$ j is persistent. $\Rightarrow$ $\ell_0 = 1$.

Therefore, we obtain

$$1 = \ell_0 p_{jj}^{(n)} + \ell_1 p_{jj}^{(n-1)} + \cdots + \ell_{n-1} p_{jj}^{(1)} + \ell_n p_{jj}^{(0)}.$$

$$\Rightarrow \quad 1 \geq \left(\ell_0 p_{jj}^{(n)} + \cdots + \overset{X_{nk}}{\boxed{\ell_k p_{jj}^{(n-k)}}}\right) \xrightarrow[n \to +\infty]{} (\ell_0 + \cdots + \ell_k) u.$$

By assumption. $u > 0$. $\Rightarrow$ $\sum_{i=0}^{k} \ell_i \leq \frac{1}{u} < +\infty$.

$\Rightarrow$ $\sum_{k=0}^{+\infty} \ell_k < +\infty$. $\Rightarrow$ $\mu_j = \sum_{k=0}^{\infty} \ell_k < +\infty$. This proves

④

$u > 0 \Rightarrow \mu_j < +\infty.$

Now let's prove the other direction, namely $\boxed{\mu_j < \infty \Rightarrow u > 0}$.

Write $X_{nk} = f_k P_{jj}^{(n-k)}$, $k = 0, \cdots, n$.

Set $X_{nk} \equiv 0$ if $k \geq n+1$.

Then. $0 \leq X_{nk} \leq f_k$. and $\lim_n X_{nk} = f_k u$.

If $\mu_j < +\infty$, then $\underbrace{\sum_k f_k}_{\substack{\| \\ \mu_j \\ \wedge \\ +\infty}} < +\infty$. Then it follows by the M-test

$$1 = \sum_{k=0}^{+\infty} X_{nk} \Rightarrow \overset{\substack{1 \\ \|}}{\lim_{n\to\infty} 1} = \lim_{n\to\infty} \sum_{k=0}^{+\infty} X_{nk}$$

$$\overset{\text{M-test}}{=} \sum_{k=0}^{+\infty} \lim_{n\to\infty} X_{nk}$$

$$= \underbrace{\sum_{k=0}^{+\infty} f_k u}_{u\mu_j}$$

$$= u \cdot \mu_j$$

$$\Rightarrow u = \frac{1}{\mu_j} \in (0, +\infty).$$

▨.

Consider an irreducible, aperiodic, and persistent chain. ⑤

Two possible cases:

① $\exists$ a stationary distribution.

(Thm 8.6 $\Rightarrow$) $\lim_{n\to\infty} p_{jj}^{(n)} = \pi_j > 0 \sim$ stationary dist.

$\sim$ positive persistent.

(By Lemma 3) $\pi_j = \frac{1}{\mu_j}$ or $\mu_j = \frac{1}{\pi_j} < +\infty$

(mean return time is finite)

② There is no stationary distn.

(Thm 8.7 $\Rightarrow$) $\lim_n p_{jj}^{(n)} = 0$. then by Lemma 3, $\mu_j = +\infty$.

$\sim$ Null persistent

Thm 8.8 Complete classification

For an irreducible and aperiodic chain, there are three cases:

1. The chain is transient.

$$\forall i, j, \ \lim_n p_{ij}^{(n)} = 0 \ \text{and} \ \sum_n p_{ij}^{(n)} < +\infty.$$

2. The chain is persistent but there is no stationary distr.

i.e., the chain is null persistent.

$$\forall i, j, \ \lim_n p_{ij}^{(n)} = 0 \ \text{but} \ \sum_n p_{ij}^{(n)} = +\infty \ \text{and} \ \mu_j = +\infty.$$

3. The chain has a stationary distr. π_j. (hence, positive persistent.)

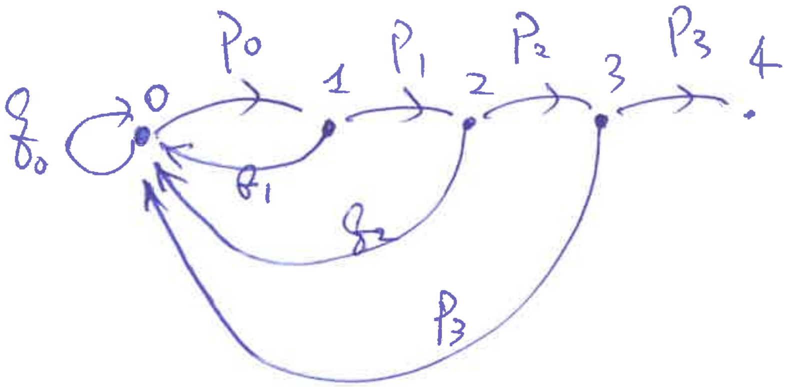
$$\forall i, j, \ \lim_n p_{ij}^{(n)} = \pi_j > 0. \ \text{and} \ \mu_j = \frac{1}{\pi_j} < +\infty.$$

Example 8.13.

$$
\begin{array}{c|c}
 & \begin{matrix} 0 & 1 & 2 & 3 & 4 & \cdots \end{matrix} \\
\begin{matrix} 0 \\ 1 \\ 2 \\ 3 \\ \vdots \end{matrix} &
\left[\begin{array}{c|ccccc}
q_0 & p_0 & & & & \cdots \\
\hline
q_1 & 0 & p_1 & & & \cdots \\
q_2 & 0 & 0 & p_2 & & \cdots \\
q_3 & 0 & 0 & 0 & p_3 & \cdots \\
\vdots & \vdots & & & & \ddots
\end{array}\right]
\end{array}
$$

i_0 zero removed.

Q substochastic matrix..



Claim: the chain is irreducible & aperiodic. ✓

Now Let ~~us~~ test if $i_0 = 0$ is transient or NOT.

Recall, in Thm 8.5. we have iff condition for this test.

The linear system associated to Q is:

$$\begin{cases} x_k = p_k \cdot x_{k+1}, & k = 1, 2, \dots \quad \text{---} (☆) \\ 0 \le x_k \le 1. & k = 1, 2 \dots \end{cases}$$

$$\boxed{\text{Suppose. } \lim_{n} \prod_{i=1}^{n} p_i = \alpha > 0.}$$

$$x_1 = p_1 \cancel{x_2}.$$

$$\cancel{x_2} = p_2 \cancel{x_3}$$

$$\cancel{x_3} = p_3 x_4$$

$$\vdots$$

$$\cancel{x_k} = p_k x_{k+1}$$

$$\Rightarrow x_1 = p_1 \cdot p_2 \cdots p_k \cdot x_{k+1}$$

$$\Rightarrow x_{k+1} = \frac{x_1}{p_1 \cdots p_k}.$$

In otherwords, we find out a nontrivial sol. to ☆.

Therefore by Thm 8.5. the chain is ~~persi~~ transient.

$f_{00} = 1-\alpha$. the chain is persistent iff $\alpha = 0$.

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