Auburn Math 7820 Lecture 10/02/2025 (1)

Continue with example 8.13

$\alpha = \prod_{i=1}^{\infty} P_i$: jumping forever to the right. or Never returns to 0 state.

Starting from 0 state.

$\Rightarrow$ $1-\alpha$ is the prob. of starting from 0 state, and eventual returning to 0 state, i.e., $f_{00} = \sum_{n=1}^{+\infty} f_{00}^{(n)}$

By definition of persistence, 0 is persistent iff

$\Leftrightarrow$ j being persistent $\Leftrightarrow f_{jj}=1$.

j being transient $\Leftrightarrow f_{jj} < 1$

$\Leftrightarrow$ $f_{00} = 1-\alpha = 1$ $\Leftrightarrow \alpha = 0$.

The sol. to ⊛ on P.7 of Lecture 9.30. takes the form

$$X_{k+1} = \frac{x_1}{p_1 \cdots p_k} = \frac{x_0}{p_0 \cdots\cdots p_k}$$

$\pi_k = \pi_0 \cdot p_0 \cdots p_{k-1}$

If π_k is the stationary dist., then

$$(\cancel{\pi_0, \pi_1, \pi_k}) \begin{pmatrix} q_0 & p_0 & \\ q_1 & & p_1 \\ \vdots & & \\ q_2 & & \end{pmatrix} = (\square, p_0\pi_0, p_1\pi_1, \cdots)$$

$$\pi_k = \pi_{k-1} \cdot p_{k-1} \quad k = 1, 2, \ldots$$

$$\pi_1 = p_0 \pi_0$$

$$\pi_2 = p_1 \pi_1$$

$$\pi_3 = p_2 \pi_2$$

$$\vdots$$

$$\pi_k = p_{k-1} \pi_{k-1}$$

$$\boxed{\pi_k = \pi_0 \times p_0 \times \cdots \times p_{k-1}}$$

for π_k to be a stationary distr., we need

$$\sum_{k=0}^{+\infty} \pi_k < +\infty \quad \text{Namely} \quad \sum_{k=0}^{+\infty} p_0 \times \cdots p_k < +\infty.$$

If this is case, $\beta := \sum_{k=0}^{\infty} p_0 \cdots p_k$ then

$$\sum_{k=}^{\infty} \pi_0 \times p_0 \cdots p_k = \pi_0 . \beta = 1.$$

$$\Rightarrow \pi_0 = \frac{1}{\beta}. \quad \text{and} \quad \pi_k = \frac{1}{\beta} p_0 \times \cdots \times p_k = \frac{p_0 \times \cdots \times p_k}{\sum_{j=0}^{\infty} p_0 \times \cdots \times p_j}$$

This is the ~~positive persistent case~~ ~~transient case~~ positive persistent case.

If $\alpha = 0$ namely $\prod_{k=0}^{n} p_0 \times \cdots p_k \to 0$ $n \to \infty$

but $\sum_{k=0}^{+\infty} p_0 \times \cdots p_k = +\infty$, then. this is the case of

null persistent.

$\Rightarrow$ Null pers.

Example: $\begin{cases} q_k = \frac{1}{k} \Rightarrow \alpha = 0. \Rightarrow \sum_{k=2}^{+\infty} p_2 \times \cdots p_k = \sum_k \frac{1}{k} = +\infty. \\ q_k = \frac{1}{k^2} \Rightarrow \alpha > 0. \Rightarrow \text{transient}. \end{cases}$

$\left(\prod_{j=2}^{+\infty} p_j = \frac{1}{2}\right)$

~~First case. (transient).~~

$\cancel{\pi_0 p = \sum_{k\geq 1}^{+\infty} p_0 \times \cdots p_k < +\infty}$

More example for transient:

$q_k = 1 - \cos\left(\frac{\pi}{2^{k+2}}\right)$ then

$p_k = \cos\left(\frac{\pi}{2^{k+2}}\right)$ and

$$\alpha = \prod_{k=0}^{+\infty} p_k = \prod_{k=0}^{+\infty} \cos\left(\frac{\pi}{2^{k+2}}\right)$$

$$= \prod_{k=1}^{+\infty} \cos\left(\frac{\pi}{2^{k+1}}\right)$$

$$= \frac{\sin\left(\frac{\pi}{2}\right)}{\pi/2} = \frac{2}{\pi} < 1$$

See Prob. 1.7 of Billingsley on P.16.