Lecture Notes for MATH 7820/7830:
—Applied Stochastic Processes I/II
(2025 Fall & 2026 Spring)

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August 28, 2025

Contents

Preface

These notes are for the course MATH 7820/7830: Applied Stochastic Processes, I/II, taught at Auburn University.

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This course sequence covers Markov chains, Markov processes, optimal stopping, martingales, renewal processes, Brownian motion, and stochastic calculus, along with their applications.

We will use and follow the textbook "**Introduction to Stochastic Processes**" by Gregory F. Lawler, Second Edition [1]. In these notes, we provide supplementary explanations and additional commentary to complement the material presented in the textbook. More materials will be provided on the course website at

https://webhome.auburn.edu/lzc0090/teaching/2025_Fall_Math7820/

Part I

Math 7820

Preliminaries

p:preliminaries⟩?

Finite Markov Chains

⟨chap:finite_mc⟩?

0.1 Example 1 of §1.3

The analysis is performed on the following 5×5 matrix $\mathbf{P}$:

$$\mathbf{P} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 1/2 & 0 & 1/2 & 0 & 0 \\ 0 & 1/2 & 0 & 1/2 & 0 \\ 0 & 0 & 1/2 & 0 & 1/2 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

This matrix represents the transition probabilities of a simple random walk on a path with 5 states.

0.1.1 Computing $\mathbf{P}^n$

Finding the Eigenvalues The eigenvalues λ are the roots of the characteristic equation $\det(\mathbf{P} - \lambda\mathbf{I}) = 0$. The characteristic polynomial for this matrix is:

$$-\lambda^5 + \frac{3}{2}\lambda^3 - \frac{1}{2}\lambda = 0$$

Factoring this polynomial gives:

$$\lambda(2\lambda^4 - 3\lambda^2 + 1) = 0 \implies \lambda(2\lambda^2 - 1)(\lambda^2 - 1) = 0$$

Solving for λ, we find the five distinct eigenvalues:

$$\lambda_1 = 1, \quad \lambda_2 = -1, \quad \lambda_3 = \frac{\sqrt{2}}{2}, \quad \lambda_4 = -\frac{\sqrt{2}}{2}, \quad \lambda_5 = 0$$

Finding the Eigenvectors For each eigenvalue λ, we find the corresponding right eigenvector $\mathbf{v}$ by solving the system of linear equations $(\mathbf{P} - \lambda\mathbf{I})\mathbf{v} = \mathbf{0}$.

Eigenvalue-Eigenvector Pairs

- For $\lambda_1 = 1$:

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

- For $\lambda_2 = -1$:

$$\mathbf{v}_2 = \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \\ 1 \end{bmatrix}$$

- For $\lambda_3 = 0$: (Note: The order of eigenvalues has been adjusted for the final decomposition matrix).

$$\mathbf{v}_3 = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \\ 1 \end{bmatrix}$$

- For $\lambda_4 = \frac{\sqrt{2}}{2}$:

$$\mathbf{v}_4 = \begin{bmatrix} \sqrt{2} \\ 1 \\ 0 \\ -1 \\ -\sqrt{2} \end{bmatrix}$$

- For $\lambda_5 = -\frac{\sqrt{2}}{2}$:

$$\mathbf{v}_5 = \begin{bmatrix} \sqrt{2} \\ -1 \\ 0 \\ 1 \\ -\sqrt{2} \end{bmatrix}$$

The eigen decomposition of a matrix is given by the formula $\mathbf{P} = \mathbf{Q}\mathbf{\Lambda}\mathbf{Q}^{-1}$. The matrix $\mathbf{Q}$ is formed by using the eigenvectors as its columns.

$$\mathbf{Q} = \begin{bmatrix} 1 & 1 & 1 & \sqrt{2} & \sqrt{2} \\ 1 & -1 & 0 & 1 & -1 \\ 1 & 1 & -1 & 0 & 0 \\ 1 & -1 & 0 & -1 & 1 \\ 1 & 1 & 1 & -\sqrt{2} & -\sqrt{2} \end{bmatrix}.$$

The matrix $\mathbf{\Lambda}$ is a diagonal matrix containing the eigenvalues corresponding to the columns of $\mathbf{Q}$.

$$\mathbf{\Lambda} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{\sqrt{2}}{2} & 0 \\ 0 & 0 & 0 & 0 & -\frac{\sqrt{2}}{2} \end{bmatrix}$$

The inverse matrix $\mathbf{Q}^{-1}$ is found by calculating the normalized left eigenvectors of $\mathbf{P}$.

$$\mathbf{Q}^{-1} = \begin{bmatrix} \frac{1}{8} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{8} \\ \frac{1}{8} & -\frac{1}{4} & \frac{1}{4} & -\frac{1}{4} & \frac{1}{8} \\ \frac{1}{4} & 0 & -\frac{1}{2} & 0 & \frac{1}{4} \\ \frac{\sqrt{2}}{8} & \frac{1}{4} & 0 & -\frac{1}{4} & -\frac{\sqrt{2}}{8} \\ \frac{\sqrt{2}}{8} & -\frac{1}{4} & 0 & \frac{1}{4} & -\frac{\sqrt{2}}{8} \end{bmatrix}$$

Multiplying these three matrices in the order $\mathbf{Q\Lambda Q}^{-1}$ will yield the original matrix $\mathbf{P}$.

Therefore, we have

$$\begin{aligned} \lim_{n\to\infty} \mathbf{P}^n &= \lim_{n\to\infty} \mathbf{Q\Lambda}^n\mathbf{Q}^{-1} \\ &= \mathbf{Q} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & (-1)^n & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \mathbf{Q}^{-1} \end{aligned}$$

This can be written as

$$\mathbf{P}^n = \mathbf{v}_1\mathbf{w}_1^\top + (-1)^n\mathbf{v}_2\mathbf{w}_2^\top,$$

where $\mathbf{v}_1 = [1, 1, 1, 1, 1]^\top$, $\mathbf{v}_2 = [1, -1, 1, -1, 1]^\top$, and

$$\mathbf{w}_1^\top = \left[\frac{1}{8}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{8}\right], \qquad \mathbf{w}_2^\top = \left[\frac{1}{8}, -\frac{1}{4}, \frac{1}{4}, -\frac{1}{4}, \frac{1}{8}\right].$$

Therefore, the limit $\lim_{n\to\infty} \mathbf{P}^n$ does not exist because of the oscillating $(-1)^n$ term. However, the even and odd subsequences converge:

- For $n = 2k$ (even powers),

$$\mathbf{P}^{2k} \to \Pi_{\text{even}} = \mathbf{v}_1\mathbf{w}_1^\top + \mathbf{v}_2\mathbf{w}_2^\top = \begin{bmatrix} \frac{1}{4} & 0 & \frac{1}{2} & 0 & \frac{1}{4} \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ \frac{1}{4} & 0 & \frac{1}{2} & 0 & \frac{1}{4} \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ \frac{1}{4} & 0 & \frac{1}{2} & 0 & \frac{1}{4} \end{bmatrix}.$$

- For $n = 2k + 1$ (odd powers),

$$\mathbf{P}^{2k+1} \to \Pi_{\text{odd}} = \mathbf{v}_1\mathbf{w}_1^\top - \mathbf{v}_2\mathbf{w}_2^\top = \begin{bmatrix} 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ \frac{1}{4} & 0 & \frac{1}{2} & 0 & \frac{1}{4} \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ \frac{1}{4} & 0 & \frac{1}{2} & 0 & \frac{1}{4} \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 \end{bmatrix}.$$

Hence, the chain has period 2.

0.1.2 Stationary Distribution

Question 0.1.1. *Find the stationary distribution* $\boldsymbol{\pi}$ *of the Markov chain:*

Step 1: The stationary distribution $\boldsymbol{\pi}$ of a Markov chain with transition matrix $\mathbf{P}$ is a row vector satisfying

$$\boldsymbol{\pi}\mathbf{P} = \boldsymbol{\pi}, \qquad \sum_{i=1}^{5} \pi_i = 1.$$

That is, $\boldsymbol{\pi}$ is a left eigenvector of $\mathbf{P}$ with eigenvalue 1, normalized to sum to 1.

Step 2: From the eigenvector decomposition, the right eigenvector for $\lambda = 1$ is $\mathbf{v}_1 = [1, 1, 1, 1, 1]^\top$. The corresponding left eigenvector (row vector) is the first row of $\mathbf{Q}^{-1}$:

$$\mathbf{w}_1^\top = \left[\tfrac{1}{8}, \tfrac{1}{4}, \tfrac{1}{4}, \tfrac{1}{4}, \tfrac{1}{8}\right].$$

This vector is already normalized to sum to 1:

$$\tfrac{1}{8} + \tfrac{1}{4} + \tfrac{1}{4} + \tfrac{1}{4} + \tfrac{1}{8} = \tfrac{1+2+2+2+1}{8} = \tfrac{8}{8} = 1.$$

Step 3: Therefore, the stationary distribution is

$$\boldsymbol{\pi} = \left[\tfrac{1}{8}, \tfrac{1}{4}, \tfrac{1}{4}, \tfrac{1}{4}, \tfrac{1}{8}\right].$$

Step 4: Interpretation: Since the chain is periodic with period 2, the stationary distribution is not the limit of $\mathbf{P}^n$ as $n \to \infty$, but it is the unique solution to $\boldsymbol{\pi}\mathbf{P} = \boldsymbol{\pi}$.

Final Answer:

$$\boxed{\boldsymbol{\pi} = \left[\tfrac{1}{8},\ \tfrac{1}{4},\ \tfrac{1}{4},\ \tfrac{1}{4},\ \tfrac{1}{8}\right]}$$

is the stationary distribution of $\mathbf{P}$.

Remark 0.1.1.

- The stationary distribution is found as the normalized left eigenvector for eigenvalue 1.
- The chain is periodic (period 2), so $\mathbf{P}^n$ does not converge, but the stationary distribution still exists and is unique.
- The stationary distribution assigns probability 1/8 to the endpoints and 1/4 to the interior states, reflecting the higher likelihood of being in the middle states in the long run.
- The answer is consistent with the eigen-decomposition and the structure of the transition matrix.

0.1.3 Cesàro Average

For a Markov chain with transition matrix $\mathbf{P}$ and stationary distribution $\boldsymbol{\pi}$, the Cesàro average of the transition matrices is defined as

$$\mathbf{A}_n = \frac{1}{n} \sum_{k=0}^{n-1} \mathbf{P}^k.$$

The Cesàro average describes the average behavior of the chain over time.

If the chain is irreducible and aperiodic, then $\mathbf{P}^n$ converges to a rank-one matrix whose rows are all $\boldsymbol{\pi}$. However, if the chain is periodic (as in this case, with period 2), $\mathbf{P}^n$ does not converge, but the Cesàro average $\mathbf{A}_n$ still converges as $n \to \infty$.

In the periodic case, the Cesàro average converges to the stationary projection:

$$\lim_{n\to\infty} \mathbf{A}_n = \mathbf{1}\boldsymbol{\pi},$$

where $\mathbf{1}$ is the column vector of all ones, so every row of the limiting matrix equals $\boldsymbol{\pi}$.

Question 0.1.2. *Find the limit of the Cesàro average of the Markov chain.*

In the example, the Cesàro average is given by

$$\frac{1}{2}\big(\Pi_{\text{even}} + \Pi_{\text{odd}}\big) = \mathbf{v}_1\mathbf{w}_1^\top = \begin{bmatrix} \frac{1}{8} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{8} \\ \frac{1}{8} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{8} \\ \frac{1}{8} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{8} \\ \frac{1}{8} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{8} \\ \frac{1}{8} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{8} \end{bmatrix},$$

so every row equals the stationary distribution

$$\boldsymbol{\pi} = \left(\tfrac{1}{8}, \tfrac{1}{4}, \tfrac{1}{4}, \tfrac{1}{4}, \tfrac{1}{8}\right).$$

Thus, the Cesàro average projects any initial distribution onto the stationary distribution in the long run.

0.2 Example 2 of §1.3

The analysis is performed on the following 5×5 transition matrix $\mathbf{P}$:

$$\mathbf{P} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1/2 & 0 & 1/2 & 0 & 0 \\ 0 & 1/2 & 0 & 1/2 & 0 \\ 0 & 0 & 1/2 & 0 & 1/2 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

This matrix represents the transition probabilities of a simple random walk on a path with 5 states, with states 1 and 5 absorbing.

0.2.1 Eigen Decomposition

Step 1: Eigenvalues. The absorbing states 1 and 5 guarantee that 1 is an eigenvalue of multiplicity at least 2. The middle 3×3 block is

$$\mathbf{A} = \begin{bmatrix} 0 & 1/2 & 0 \\ 1/2 & 0 & 1/2 \\ 0 & 1/2 & 0 \end{bmatrix},$$

with characteristic polynomial

$$\det(\mathbf{A} - \lambda I) = -\lambda^3 + \tfrac{1}{2}\lambda.$$

Thus, $\lambda = 0,\ \pm 1/\sqrt{2}$ are the eigenvalues of $\mathbf{A}$.

Hence, the eigenvalues of $\mathbf{P}$ are

$$\lambda_1 = 1, \quad \lambda_2 = 1, \quad \lambda_3 = -\tfrac{1}{\sqrt{2}}, \quad \lambda_4 = \tfrac{1}{\sqrt{2}}, \quad \lambda_5 = 0.$$

Step 2: Eigenvectors. With the chosen ordering, the corresponding right eigenvectors are

$$\lambda = 1: \qquad \mathbf{v}_1 = \begin{bmatrix} -3 \\ -2 \\ -1 \\ 0 \\ 1 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 4 \\ 3 \\ 2 \\ 1 \\ 0 \end{bmatrix};$$

$$\lambda = -\tfrac{1}{\sqrt{2}}: \quad \mathbf{v}_3 = \begin{bmatrix} 0 \\ 1 \\ -\sqrt{2} \\ 1 \\ 0 \end{bmatrix};$$

$$\lambda = \tfrac{1}{\sqrt{2}}: \qquad \mathbf{v}_4 = \begin{bmatrix} 0 \\ 1 \\ \sqrt{2} \\ 1 \\ 0 \end{bmatrix};$$

$$\lambda = 0: \qquad \mathbf{v}_5 = \begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}.$$

Step 3: Decomposition. Collecting these eigenvectors as columns gives

$$\mathbf{V} = \begin{bmatrix} -3 & 4 & 0 & 0 & 0 \\ -2 & 3 & 1 & 1 & -1 \\ -1 & 2 & -\sqrt{2} & \sqrt{2} & 0 \\ 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix},$$

with inverse

$$\mathbf{V}^{-1} = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ \frac{1}{4} & 0 & 0 & 0 & \frac{3}{4} \\ \frac{-2+\sqrt{2}}{8} & \frac{1}{4} & -\frac{1}{2\sqrt{2}} & \frac{1}{4} & \frac{-2+\sqrt{2}}{8} \\ \frac{-2-\sqrt{2}}{8} & \frac{1}{4} & \frac{1}{2\sqrt{2}} & \frac{1}{4} & \frac{-2-\sqrt{2}}{8} \\ \frac{1}{4} & -\frac{1}{2} & 0 & \frac{1}{2} & -\frac{1}{4} \end{bmatrix}.$$

Then

$$\mathbf{P} = \mathbf{V}\,\boldsymbol{\Lambda}\,\mathbf{V}^{-1}, \qquad \boldsymbol{\Lambda} = \operatorname{diag}\Big(1,\, 1,\, -\tfrac{1}{\sqrt{2}},\, \tfrac{1}{\sqrt{2}},\, 0\Big).$$

0.2.2 Asymptotic Behavior of $\mathbf{P}^n$

For $n \geq 1$,

$$\mathbf{P}^n = \mathbf{V}\,\boldsymbol{\Lambda}^n\,\mathbf{V}^{-1}.$$

The contributions from eigenvalues $0, \pm 1/\sqrt{2}$ vanish as $n \to \infty$. Thus

$$\lim_{n\to\infty} \mathbf{P}^n = \mathbf{U} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ \frac{3}{4} & 0 & 0 & 0 & \frac{1}{4} \\ \frac{1}{2} & 0 & 0 & 0 & \frac{1}{2} \\ \frac{1}{4} & 0 & 0 & 0 & \frac{3}{4} \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Thus $\mathbf{U}$ encodes absorption probabilities:

$$\mathbb{P}\left(\text{absorption at } 1 \mid X_0 = i\right) = \frac{5-i}{4}, \qquad \mathbb{P}\left(\text{absorption at } 5 \mid X_0 = i\right) = \frac{i-1}{4}, \qquad i = 1, \ldots, 5.$$

0.2.3 Stationary Distributions

Since $\mathbf{P}$ is absorbing, the set of stationary distributions is

$$\pi = \alpha\,\mathbf{e}_1 + (1-\alpha)\,\mathbf{e}_5, \quad 0 \le \alpha \le 1,$$

where $\mathbf{e}_1 = (1,0,0,0,0)$, $\mathbf{e}_5 = (0,0,0,0,1)$.

For any initial distribution μ,

$$\mu\mathbf{P}^n \xrightarrow[n\to\infty]{} \mu\mathbf{U},$$

which is a convex combination of $\mathbf{e}_1$ and $\mathbf{e}_5$ with weights given by the absorption probabilities.

0.2.4 Cesàro Average

Finally, consider the Cesàro average:

$$\frac{1}{n}\sum_{k=0}^{n-1} \mathbf{P}^k.$$

As $n \to \infty$, contributions from non-unit eigenvalues vanish, leaving the projection onto the $\lambda = 1$ eigenspace. Thus

$$\frac{1}{n}\sum_{k=0}^{n-1} \mathbf{P}^k \xrightarrow[n\to\infty]{} \mathbf{U}.$$

Hence both $\mathbf{P}^n$ and the Cesàro average converge to the same absorbing-probability projector $\mathbf{U}$.

Countable Markov Chains

ap:countable_mc⟩?

Continuous-Time Markov Chains

p:continuous_mc⟩?

Optimal Stopping

ptimal_stopping⟩?

Martingales

hap:martingales⟩?

Part II

Math 7830

Renewal Processes

newal_processes⟩?

Reversible Markov Chains

p:reversible_mc⟩?

Brownian Motion

brownian_motion⟩?

Stochastic Integration

tic_integration⟩?

Bibliography

[1] Gregory F. Lawler, *Introduction to stochastic processes*, Second, Chapman & Hall/CRC, Boca Raton, FL, 2006. MR2255511.