Abstract. Given an arbitrary nondegenerate metric continuum \( P \), we give an uncountable family of topologically distinct metric compactifications of \( [1, \infty) \), having \( P \) as the remainder.

1. Introduction

All spaces considered in this paper are metric. A continuum is a nondegenerate compact connected metric space. Two continua are incomparable if neither of them is the continuous image of the other. By the ray we understand any space homeomorphic to \([1, \infty)\). We say that a continuum \( X \) is a compactification of the ray if \( X \) can be represented as the union of the ray \( R \) and a continuum \( P \) such that \( R \cap P = \emptyset \) and \( P = \text{cl}(R) \setminus R \). If \( X \) is a compactification of the ray, then the representation \( X = R \cup P \) is unique. The continuum \( P \) is called the remainder of the compactification.

Metric compactifications of the ray have been widely researched by many authors. In 1932 Waraszkiewicz [14] constructed an uncountable collection of mutually incomparable compactifications of the ray each with a simple closed curve as the remainder, see also [11] for a simpler proof.

In [12] Rogers raised the question: Does there exist an uncountable collection of mutually incomparable chainable continua? In [4] Bellamy gave an affirmative answer to this question. Each member of Bellamy’s collection had infinitely many path components.

In [10, Lemma 5.1, p.20] Nadler and Quinn proved that every metric compactification of the ray with the arc as remainder, is embeddable in the Euclidean plane \( \mathbb{R}^2 \) with the arc as the convex vertical segment \( \{1\} \times [0, 1] \) and the ray as the graphic of a continuous map defined on the interval \([0, 1)\).

Awartani used [10, Lemma 5.1, p.20] to prove the main results in [1], [2] and [3]. In particular in [3] he proved the existence of an uncountable collection of incomparable chainable continua within the family of compactifications of the ray with the arc as remainder.

In 1995 Awaartani asked if all metric compactifications of the ray with remainder being the pseudo-arc were homeomorphic. This question was answered in the

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negative in [6], then in [7] an uncountable family of such compactifications is constructed. We generalize this last result by replacing the pseudo-arc by an arbitrary continuum. We prove (see Theorem 1) that given any (metric and nondegenerate) continuum \( P \), there exists an uncountable family of non homeomorphic metric compactifications of the ray, having \( P \) as the remainder.

As we mentioned above, the continua constructed in [3], [4], [11] and [14] were not only topologically distinct, but also mutually incomparable. The same is true for many other constructions of the similar type, see for example [5], [8], [9] and [13]. However, continua constructed in the proof of Theorem 1 are only not homeomorphic but some of them could be mapped onto some others. Therefore, it would be interesting to answer the following question.

**Question 1.** Given any (metric and nondegenerate) continuum \( P \), does there exist an uncountable family of non comparable metric compactifications of the ray, having \( P \) as the remainder? In particular, what if \( P \) is the pseudo-arc?

## 2. The result

Let \( Q \) denote the Hilbert cube and let \( d \) be the distance on \( Q \). Let \( H_d \) denote the Hausdorff distance induced by \( d \) on the set of nonempty closed subsets of \( Q \). Let \( Q_1 \) denote \( Q \times [0,1] \) and let \( d_1 \) denote the distance on \( Q_1 \) given by \( d_1((x,t),(y,s)) = d(x,y) + |t-s| \).

**Theorem 1.** For each nondegenerate continuum \( P \) there is uncountably many topologically distinct compactifications of \([1, \infty)\) each with \( P \) as the remainder.

**Proof.** Let \( P \) be a nondegenerate continuum. Since each metric continuum is separable, it can be embedded in the Hilbert cube. Therefore, we can assume without loss of generality that \( P \subset Q \times \{0\} \subset Q_1 \).

Let \( \mathcal{X} \) denote the collection of all subcontinua of \( Q_1 \) which are compactifications of \([1, \infty)\) with \( P \) as the remainder.

Suppose \( \mathcal{X}_0 \) is a countable subcollection of \( \mathcal{X} \). To prove the theorem, it is enough to construct \( Y \subset \mathcal{X} \) such that \( Y \) is not homeomorphic to any element of \( \mathcal{X}_0 \).

Set \( \alpha = \tfrac{1}{2}\operatorname{diam}(P) \). Let \( P' \subset Q \) denote the projection of \( P \subset Q \times \{0\} \) into \( Q \). Let \( S \) be a nondegenerate proper subcontinuum of \( P' \) and let \( U \) be a neighborhood of \( S \) in \( Q \) such that \( P' \setminus \text{cl}(U) \neq \emptyset \). Take a point \( v \in P' \setminus \text{cl}(U) \). Let \( \beta < \diam(S) \) be a positive number such that \( d(u,v) > \beta \) for each \( u \in \text{cl}(U) \).

Let \( E \) denote the set of positive rational numbers less than \( \alpha \). Consider the countable set \( \mathcal{X}_0 \times E \) and arrange its elements into a sequence \( \{(X_n,\epsilon_n): n \in \mathbb{N}\} \) where \( \mathbb{N} \) denotes the set of positive integers. For each \( n \in \mathbb{N} \), let \( R_n \) denote \( X_n \setminus P \) and let \( \epsilon_n \) denote the endpoint of \( R_n \). For any two points \( a \) and \( b \) in \( R_n \), let \( [a,b]_n \) denote the the arc in \( R_n \) with its endpoints being \( a \) and \( b \).

We will construct a continuous function \( g : [1, \infty) \to Q \), and consider the embedding \( f : [1, \infty) \to Q \times [0,1] = Q_1 \) given by \( f(t) = (g(t),\tfrac{1}{t}) \). We will then define \( Y \) as the closure of the image of \( f \).

We will construct \( g \) on on intervals of the form \([n, n+1]\) by induction with respect to \( n \) so that the following three conditions are satisfied.

1. \( H_d(g([n, n+1]), P') < \frac{1}{n} \), (where \( H_d \) is the Hausdorff distance induced by \( d \)).
2. \( g(n+1) \in S \).
(3) For each embedding \( h \) of \( X_n \) into \( Q_1 \) such that \( h(P) = P \) and \( f([1, n + 1]) \subset h(R_n) \), we have that either

- \( d_1(h(a_0), h(a_1)) < \epsilon_n \) for some \( a_0, a_1 \in X_n \) such that \( d_1(a_0, a_1) > \alpha \),

or

- \( d_1(h(b_0), h(b_1)) > \beta \) for some \( b_0, b_1 \in X_n \) such that \( d_1(b_0, b_1) < \epsilon_n \).

Set \( g(1) \) to be any point of \( S \). Suppose that \( g \) has been constructed on the interval \([1, n] \). We will now construct \( g \) on \([n, n + 1] \).

Since \( g \) has been defined on \([1, n] \), \( f \) has also been defined on \([1, n] \). Thus, \( f([1, n]) \) is either an arc or a point (if \( n = 1 \)).

**Observation 1.** There is a positive integer \( k \) such that there is no collection of \( k \) mutually disjoint arcs contained in \( f([1, n]) \), the diameter of each greater than or equal to \( \epsilon_n \).

**Observation 2.** Since \( \text{diam}(P) = 2\alpha \) and \( cl(R_n) \setminus R_n = P \), there is a point \( w \in R_n \) such that \([\epsilon_n, w]_n \) contains a collection \( A \) of \( k \) mutually disjoint arcs each with the diameter greater than \( \alpha \).

**Observation 3.** Since \( cl(R_n) \setminus R_n = P \), there exists a point \( z \in R_n \setminus [\epsilon_n, w]_n \) such that for each \( p \in P \) there is \( q \in [w, z]_n \) such that \( d_1(p, q) < \epsilon_n \).

**Observation 4.** There is a positive integer \( m \) such that there is no collection of \( m \) mutually disjoint arcs contained in \([\epsilon_n, z]_n \), the diameter of each greater than or equal to \( \epsilon_n \).

Let \( N_n \) denote \( 1/n \)-neighborhood of \( S \) in \( Q \). Set \( C = U \cap N_n \). Extend \( g \) to \([n, n + 0.5] \) so that \( g([n, n + 0.5]) \subset C \) and there is a collection \( M_0 \) of \( m \) mutually disjoint arcs contained in \([n, n + 0.5] \) such that \( \text{diam}(g(M)) > \beta \) for each \( M \in M_0 \), (this is possible since \( \beta < \text{diam}(S) \)). Set \( M = \{ f(M) | M \in M_0 \} \). Since \( \text{diam}(g(M)) \leq \text{diam}(f(M)) \) the following observation is true.

**Observation 5.** \( M \) is a collection of \( m \) mutually disjoint arcs in \( f([n, n + 0.5]) \) such that \( \text{diam}(M) > \beta \) for each \( M \in M \).

Now extend \( g \) to \([n + 0.5, n + 1] \) so that \( g(n + 1) \in S \) and \( H_{\beta}(g([n + 0.5, n + 1]), P') < \frac{1}{n} \). Notice that the above extensions are possible because \( Q \) is locally path-connected.

We will now observe that property (3) of the construction is satisfied. Suppose to the contrary that there is an embedding \( h : X_n \to Q_1 \) such that \( h(P) = P \), \( h(\epsilon_n) = f(1) \) and \( f([1, n + 1]) \subset h(R_n) \), and the following two conditions are satisfied:

(i) \( d_1(h(a_0), h(a_1)) \geq \epsilon_n \) for all \( a_0, a_1 \in X_n \) such that \( d_1(a_0, a_1) > \alpha \).

(ii) \( d_1(h(b_0), h(b_1)) \leq \beta \) for all \( b_0, b_1 \in X_n \) such that \( d_1(b_0, b_1) < \epsilon_n \).

We will now state and prove the following two claims.

**Claim 6.** \( h(w) \notin f([1, n]) \).

Suppose to the contrary that \( h(w) \in f([1, n]) \), since \( f([1, n + 1]) \subset h(R_n) \) and \( h(\epsilon_n) = f(1) \), it follows that \( h([\epsilon_n, w]_n) \subset f([1, n]) \). By Observation 2, \([\epsilon_n, w]_n \) contains a collection \( A \) of \( k \) mutually disjoint arcs each with the diameter greater than \( \alpha \). Since the collection \( \{ h(A) | A \in A \} \) consists of \( k \) mutually disjoint arcs all contained in \( f([1, n]) \), it follows from Observation 1 that \( \text{diam}(h(A)) < \epsilon_n \) for some \( A \in A \) contrary to (i). Therefore the claim is true.
Claim 7. \( h(z) \in f([n, n + 0.5]). \)

Suppose to the contrary that \( h(z) \notin f([n, n + 0.5]). \) Then by a similar argument to the one used in the proof of Claim 6 we infer that \( f([1, n + 0.5]) \subseteq h([e_n, z]_n). \) Let \( M_1 = \{ h^{-1}(M) \mid M \in M \} \) where \( M \) is the collection from Observation 5. Clearly, \( M_1 \) is a collection of \( n \) mutually disjoint arcs in \( [e_n, z]_n. \) It follows from (ii) and Observation 5 that \( \text{diam}(M) \geq \epsilon_n \) for each \( M \in M_1. \) This contradicts Observation 4. Hence, the claim is true.

It follows from Claims 6 and 7 that \( h([w, z]_n) \subset f([n, n + 0.5]). \) Let \( p \in P \) be such that \( h(p) = (e, 0). \) By Observation 3, there exists a point \( q \in [w, z]_n \) such that \( d_1(p, q) < \epsilon_n. \) By (ii) we have that \( d_1((v, 0), h(q)) \leq \beta. \)

On the other hand, \( h(q) \in f([n, n + 0.5]), \) then there is \( t \in [n, n + 0.5] \) such that \( h(q) = (g(t), \frac{1}{4}). \) Notice that \( g(t) \in U, \) hence \( d(v, g(t)) > \beta. \) Therefore, \( d_1((v, 0), h(q)) = d_1((v, 0), (g(t), \frac{1}{4})) > d(v, g(t)) > \beta. \) This contradiction proves (3) and completes the construction of \( g. \)

Set \( Y = \text{cl}(f([1, \infty])). \) Clearly \( Y \subset X. \) To complete the proof of the theorem it is enough to prove that \( Y \) is not homeomorphic to any element of \( X_0. \) Suppose to the contrary that there is a homeomorphism \( h \) mapping some \( X \in X_0 \) onto \( Y. \) Then, there exists a positive number \( \delta \) such that \( d_1(h(a_0), h(a_1)) > \delta \) for all \( a_0, a_1 \in X \) such that \( d_1(a_0, a_1) > \alpha. \) There is a positive number \( \eta \) such that \( d_1(h(b_0), h(b_1)) < \beta \) for all \( b_0, b_1 \in X \) such that \( d_1(b_0, b_1) < \eta. \) Let \( \epsilon \) be a positive rational number less than each of the numbers \( \alpha, \delta \) and \( \eta. \) There is a positive integer \( n \) such that \( X_n = X \) and \( \epsilon_n = \epsilon. \) Observe that the condition (3) of the construction of \( g \) contradicts the choice of \( \epsilon. \) This contradiction completes the proof of the theorem.

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\text{REFERENCES}
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(Martínez-de-la-Vega) *Instituto de Matemáticas, Universidad Nacional Autónoma de México, Circuito Exterior, Cd. Universitaria, México, 04510, D.F.*

*E-mail address: vvm@matem.unam.mx*

(Minc) *Department of Mathematics and Statistics, Auburn University, Auburn, Alabama 36849, USA*

*E-mail address: mincio@auburn.edu*