NOTES FOR ARBEITSGEMEINSCHAFT: NEW HOMOLOGICAL METHODS IN TORIC GEOMETRY

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Contents

1. Lecture 1 (Amal Matoo): Derived Category of $\mathbb{P}^n$, I

A common method for studying modules is to "replace" the module by a complex of simpler objects, e.g. replacing a module by its free resolution. A similar method applies to sheaves, where one replaces a sheaf by a line bundle resotluion, or by a Čech resolution. One of the ideas behind a derived category is to treat the resolution and the original object as "the same", i.e. to have a framework where a resolution is equivalent to the object it resolves.

Unless otherwise specified, $\mathbf{k}$ denotes an algebraically closed field of arbitrary characteristic, and every variety or scheme is assumed to be over $\mathbf{k}$. We note that, throughout these notes, we provide citations to certain results–most notably those from [CLS11]–where the assumption is that $\mathbf{k}$ has characteristic 0. Generally the characteristic zero results can easily be extended to arbitrary characteristic, though there are a few exceptions where the extension is more subtle, e.g. when it comes to actions of finite groups.

1.1. What is a derived category?

Let X be a variety over a field $\mathbf{k}$, and let $\operatorname{coh} X$ denote the abelian category of coherent sheaves on X. The central object of these notes is the bounded derived category $D(X)$.[1] As noted above, the category $D(X)$ naturally "contains" all homological information carried by $\operatorname{coh} X$, but it turns out that $D(X)$ often has better properties. We briefly recall its construction and refer to [Huy06] for a thorough treatment.

Let $\mathcal{E}^\bullet$ and $\mathcal{F}^\bullet$ be bounded complexes of coherent sheaves on X. As a reminder, this means that

$$\mathcal{E}^\bullet = \cdots \mathcal{E}^i \xrightarrow{\varphi^i} \mathcal{E}^{i+1} \xrightarrow{\varphi^{i+1}} \cdots$$

is a complex such that each $\mathcal{E}^i$ is a coherent sheaf, and where $\mathcal{E}^i = 0$ for all but finitely many values of i. Let $f\colon \mathcal{E}^\bullet \to \mathcal{F}^\bullet$ be a morphism of complexes. This naturally induces morphisms

$$H^i f\colon H^i(\mathcal{E}^\bullet) \to H^i(\mathcal{F}^\bullet)$$

for all i, where $H^i(\mathcal{E}^\bullet) := \ker \varphi^i / \operatorname{im} \varphi^{i-1}$. We say that f is a quasi-isomorphism if each of these maps $H^i f$ is an isomorphism of sheaves on X.

Definition 1.1. The bounded derived category of X has objects given by bounded complexes of coherent sheaves and morphisms given by chain maps modulo homotopy, with quasi-isomorphisms formally inverted.

Remark 1.2. We are aiming to include as little discussion of the technical aspects of derived categories as possible, but there is a rich world here to be explored. For instance, by definition, a morphism is a finite "zig-zag diagram" of genuine morphisms of complexes and formally inverted quasi-isomorphisms. It turns out that a simpler description works: any morphism in the derived category $\mathcal{E} \longrightarrow \mathcal{F}$ in $D(X)$ can be represented by a roof $\mathcal{E} \xleftarrow{\simeq} \mathcal{E}' \longrightarrow \mathcal{F}$ in which the left arrow is a quasi-isomorphism. See [Căl05] or [Huy06] for excellent expository introductions.

Example 1.3. On $\mathbb{P}^2$, there are no nonzero maps $\mathcal{O} \to \mathcal{O}(-3)$ of sheaves, since $H^0(\mathbb{P}^2, \mathcal{O}(-3)) = 0$. However, the Koszul complex

$$\mathcal{O} \xleftarrow{\simeq} \left[\, \mathcal{O}(-1)^3 \longleftarrow \mathcal{O}(-2)^3 \longleftarrow \mathcal{O}(-3) \,\right]$$

is exact as a complex of sheaves on $\mathbb{P}^2$. Thus the Koszul complex induces a quasi-isomorphism between the sheaf $\mathcal{O}$ and the complex $[\mathcal{O}(-1)^3 \leftarrow \mathcal{O}(-2)^3 \leftarrow \mathcal{O}(-3)]$. The derived category allows us to formally invert this quasi-isomorphism. There is an honest morphism of complexes from $[\mathcal{O}(-1)^3 \leftarrow \mathcal{O}(-2)^3 \leftarrow \mathcal{O}(-3)]$ to the complex $\mathcal{O}(-3)[2]$, which consists of $\mathcal{O}(-3)$ in cohomological position -2. Specifically, that morphism simply sends $\mathcal{O}(-3)$ to $\mathcal{O}(-3)$ by the identity and sends

[1] Technical note: since we will never consider unbounded derived categories in these notes, we can write $D(X)$ instead of $D^b(X)$ or various other notations found in the literature.

the other sheaves to 0. Combining these, we obtain a morphism in the derived category, from $\mathcal{O}$ to $\mathcal{O}(-3)[2]$:

$$\mathcal{O} \xleftarrow{\simeq} \big[\, \mathcal{O}(-1)^3 \longleftarrow \mathcal{O}(-2)^3 \longleftarrow \mathcal{O}(-3) \,\big] \xrightarrow{\pi} \mathcal{O}(-3)[2].$$

For those more familiar with derived functors, we remark that this morphism $\mathcal{O} \to \mathcal{O}(-3)[2]$ in the derived category corresponds to an extension class in $\mathrm{Ext}^2(\mathcal{O}, \mathcal{O}(-3)) = H^2(\mathbb{P}^2, \mathcal{O}(-3)) \cong \mathbf{k}$.

Passing to the derived category destroys the abelian structure: kernels and cokernels no longer make sense, and there are no short exact sequences. What survives is the structure of a triangulated category, in which short exact sequences are replaced by distinguished triangles.

Definition 1.4. A triangulated category is an additive category $\mathcal{D}$ equipped with an autoequivalence $E \mapsto E[1]$ (the shift) and a class of distinguished triangles $E \to F \to G \to E[1]$ subject to a list of axioms (see [Huy06, §1.2]). Moreover, every morphism $f \colon E \to F$ fits into a distinguished triangle

$$E \xrightarrow{\ f\ } F \longrightarrow \mathrm{cone}(f) \longrightarrow E[1].$$

The category $D(X)$ is triangulated, with the shift given on a complex $\mathcal{E}^\bullet$ by $\mathcal{E}[1]^i = \mathcal{E}^{i+1}$ and $d_{\mathcal{E}[1]} = -d_{\mathcal{E}}$, and with $\mathrm{cone}(f)^i = \mathcal{E}^{i+1} \oplus \mathcal{F}^i$ for a chain map $f \colon \mathcal{E}^\bullet \to \mathcal{F}^\bullet$. A full subcategory closed under shifts and cones is a triangulated subcategory; it is thick if it is moreover closed under direct summands. The smallest thick triangulated subcategory containing a set of objects $\{E_i\}$ is the subcategory they generate, and we say $\{E_i\}$ generates $\mathcal{D}$ if this subcategory is all of $\mathcal{D}$. We say a functor between triangulated categories that preserves shifts and distinguished triangles is exact.

Now we come to a key point: there is a fully faithful embedding $\mathrm{coh}\, X \hookrightarrow D(X)$ sending a sheaf $\mathcal{E}$ to the complex consisting only of $\mathcal{E}$ in cohomological degree 0:

$$\mathcal{E} \mapsto [\cdots \to 0 \to \mathcal{E} \to 0 \to \cdots].$$

What "fully faithful" means is that morphisms are preserved. More is true, as *derived* morphisms are also preserved. That is, if you take two sheaves $\mathcal{E}$ and $\mathcal{F}$, then we have a natural isomorphism:

$$\mathrm{Ext}^i_X(\mathcal{E}, \mathcal{F}) \cong \mathrm{Hom}_{D(X)}(\mathcal{E}, \mathcal{F}[i]).$$

In other words, the Ext groups from the category of sheaves are realized as precisely the morphisms in the derived category, with Ext^i corresponding to morphisms from $\mathcal{E}$ to the shift $\mathcal{F}[i]$.

In big picture terms, one thing that this means is that there is no loss of information when we pass from $\mathrm{coh}\, X$ to $D(X)$. So if we can understand properties about $D(X)$, then this will tell us properties about the thing we originally cared about, which was $\mathrm{coh}\, X$. If $D(X)$ is easier to describe than $\mathrm{coh}\, X$—which is the case in many examples—then this provides motivation for studying $D(X)$ instead of $\mathrm{coh}\, X$.

We henceforth identify a sheaf with the corresponding complex so that a short exact sequence $0 \to \mathcal{E} \to \mathcal{F} \to \mathcal{G} \to 0$ in $\mathrm{coh}\, X$ is represented by a distinguished triangle $\mathcal{E} \to \mathcal{F} \to \mathcal{G} \to \mathcal{E}[1]$ in $D(X)$, but note that distinguished triangles allow extensions between objects that live in different cohomological degrees, thus not every distinguished triangle comes from a short exact sequence.

1.2. **Fourier–Mukai transforms.** Any object of the derived category of a product $X \times Y$ induces an exact functor $D(X) \to D(Y)$. The resulting functors play a major role in the study of sheaves on varieties, and will come up throughout these notes.

Most standard functors of sheaves from $\mathrm{coh}\, X$ to $\mathrm{coh}\, Y$ will induce a corresponding (total) derived functor from $D(X)$ to $D(Y)$. For instance, if $f \colon X \to Y$ is a proper morphism, then $f_* \colon \mathrm{coh}\, X \to \mathrm{coh}\, Y$ is left exact, and properness guarantees both that f_* preserves coherence and that the derived functor takes bounded complexes to bounded complexes, so that there is a corresponding functor $Rf_*\colon D(X) \to D(Y)$. We will not give a full review of derived functors, again referring to standard references for those unfamiliar with this theory.

Fourier–Mukai transforms are built as compositions of standard derived functors.

Definition 1.5. Let X and Y be smooth projective varieties, with projections

$$\begin{tikzcd} & X \times Y \arrow[dl, "p"'] \arrow[dr, "q"] & \\ X & & Y. \end{tikzcd}$$

The Fourier–Mukai transform with kernel $\mathcal{K} \in D(X \times Y)$ is the exact functor

$$\Phi_{\mathcal{K}}\colon D(X) \xrightarrow{Lp^*} D(X \times Y) \xrightarrow{-\overset{L}{\otimes}\mathcal{K}} D(X \times Y) \xrightarrow{Rq_*} D(Y),$$

where Lp^*, $-\overset{L}{\otimes}\mathcal{K}$, and Rq_* are the derived functors induced by p^*, $-\otimes\mathcal{K}$, and q_*, respectively, but since p is flat Lp^* is the usual pullback.

Remark 1.6. Those seeing this definition for the first time may have the impression that Fourier–Mukai transforms are somewhat technical, and that it would be difficult to work out an explicit example in practice. This is correct! It is easy to write out a formal definition of a Fourier–Mukai transform, and these functors do have nice properties. But in practice, these functors can be very subtle, and it can be hard to understand how to compute even seemingly simple examples. This will be a key theme in later sections. In fact, perhaps the most technical lemma in the entire notes amounts to computing the Fourier–Mukai transforms of a finite set of line bundles. (See Lecture 14.)

Remark 1.7. A theorem of Orlov (which is quite interesting, but will play no role in these notes) asserts that, for smooth projective X and Y, every exact equivalence $D(X) \xrightarrow{\simeq} D(Y)$ arises as a Fourier–Mukai transform for some kernel $\mathcal{K}$, which is unique up to isomorphism in the derived category [Orl97].

By far the most important case of the Fourier–Mukai transform occurs when $Y = X$ and $\mathcal{K} \in D(X \times X)$ is the structure sheaf $\mathcal{O}_\Delta := \iota_*\mathcal{O}_X$ of the diagonal subscheme $\iota\colon \Delta \to X \times X$. In this case we have:

Proposition 1.8. *The functor $\Phi_{\mathcal{O}_\Delta}\colon D(X) \to D(X)$ is isomorphic to the identity functor.*

Proof. Write ι for the diagonal embedding, so that $p \circ \iota = q \circ \iota = \mathrm{id}_X$. Using the projection formula, and the isomorphism $L\iota^* \circ Lp^* \cong L(p\circ\iota)^* = \mathrm{id}$ (note that ι is not flat, so the derived pullback $L\iota^*$ is essential here), we compute

$$\Phi_{\mathcal{O}_\Delta}(\mathcal{E}) = Rq_*\left(p^*\mathcal{E} \overset{L}{\otimes} \iota_*\mathcal{O}_X\right) \cong Rq_*\iota_*(L\iota^*p^*\mathcal{E}) \cong R(q\circ\iota)_*\, L(p\circ\iota)^*\mathcal{E} \cong \mathcal{E}. \qquad \square$$

1.3. **Beilinson's resolution of the diagonal.** Because we are working in the derived category, the sheaf $\mathcal{O}_\Delta$ is isomorphic to any resolution of that sheaf. Proposition 1.8 can therefore become a powerful computational tool if we can replace $\mathcal{O}_\Delta$ by an explicit and simple locally free resolution. In this case, the left derived tensor product with a complex of locally free sheaves will become the usual tensor product, allowing for more concrete applications.

On $X = \mathbb{P}^n$, exactly such a resolution was constructed by Beilinson [Bei78]. Let $\mathcal{T}_{\mathbb{P}^n}$ denote the tangent bundle and set

$$\mathcal{W} := \mathcal{O}_{\mathbb{P}^n}(1) \boxtimes \mathcal{T}_{\mathbb{P}^n}(-1),$$

a vector bundle of rank n on $\mathbb{P}^n \times \mathbb{P}^n$, where $\mathcal{F} \boxtimes \mathcal{G} := p^*\mathcal{F} \otimes q^*\mathcal{G}$ denotes the external tensor product. There is a canonical section $s \in H^0(\mathbb{P}^n \times \mathbb{P}^n, \mathcal{W})$ whose vanishing locus is precisely the diagonal Δ (see [Căl05, §3] or [Huy06, §8.3]). Since $\mathbb{P}^n \times \mathbb{P}^n$ is smooth, hence Cohen–Macaulay, and the vanishing locus Δ has codimension $n = \operatorname{rank} \mathcal{W}$, the section s is regular, so the Koszul complex on s is a locally free resolution

$$0 \to \bigwedge^n \mathcal{W}^\vee \to \cdots \to \bigwedge^2 \mathcal{W}^\vee \to \mathcal{W}^\vee \to \mathcal{O}_{\mathbb{P}^n \times \mathbb{P}^n} \to \mathcal{O}_\Delta \to 0.$$

Using $\mathcal{W}^\vee \cong \mathcal{O}(-1) \boxtimes \Omega^1_{\mathbb{P}^n}(1)$ and $\bigwedge^i \big(\Omega^1(1)\big) \cong \Omega^i(i)$, we obtain $\bigwedge^i \mathcal{W}^\vee \cong \mathcal{O}(-i) \boxtimes \Omega^i_{\mathbb{P}^n}(i)$, so Beilinson's resolution takes the form

$$\mathcal{O}_\Delta \simeq \Big[\mathcal{O}(-n) \boxtimes \Omega^n(n) \to \cdots \to \mathcal{O}(-1) \boxtimes \Omega^1(1) \to \mathcal{O} \boxtimes \mathcal{O}\Big],$$

a complex each of whose terms is a box product of a line bundle with a twisted exterior power of the cotangent bundle. In the next lecture we combine this resolution with Proposition 1.8 to prove that the line bundles $\mathcal{O}(-n), \mathcal{O}(-n+1), \ldots, \mathcal{O}$ generate $D(\mathbb{P}^n)$. The same circle of ideas—now applied to resolutions of the diagonal on toric varieties—reappears in Lecture 13 as the Fourier–Mukai transforms at the heart of the Cox category.

2. Lecture 2 (Abhik Pal): Derived Category of $\mathbb{P}^n$, II

One reason that we work with derived categories instead of the category of coherent sheaves is that derived categories often allow for more elegant global descriptions and thus cleaner structural insights into properties of sheaves. Most commonly, this is done by decomposing the derived category into simple building blocks, in a manner akin to finding a basis for a vector space.

To make this decomposition precise, we need to introduce the notions of a semiorthogonal decomposition and, in the best case, a full strong exceptional collection. See [Huy06, §1.4], [Căl05, §3], or [Kuz14] for more details.

Throughout this discussion, $\mathcal{D}$ denotes a $\mathbf{k}$-linear triangulated category, the case of interest being $\mathcal{D} = D(X)$ for a smooth projective variety X. Given a full triangulated subcategory $\mathcal{B} \subseteq \mathcal{D}$, its right orthogonal $\mathcal{B}^\perp$ is the full subcategory of objects C with $\operatorname{Hom}_{\mathcal{D}}(B, C) = 0$ for all $B \in \mathcal{B}$; the left orthogonal ${}^\perp\mathcal{B}$ is defined symmetrically. A sequence $(\mathcal{B}_0, \ldots, \mathcal{B}_r)$ of triangulated subcategories is semiorthogonal if $\mathcal{B}_j \subseteq \mathcal{B}_i^\perp$ whenever $j < i$.

Definition 2.1. A semiorthogonal decomposition $\mathcal{D} = \langle \mathcal{B}_0, \ldots, \mathcal{B}_r \rangle$ is a semiorthogonal sequence of subcategories that together generate $\mathcal{D}$. That is, the smallest thick triangulated subcategory containing all objects from every $\mathcal{B}_i$ is precisely $\mathcal{D}$.

We should think of this as akin to writing a vector space as a direct sum of subspaces. The best possible scenario would be to have a basis for the vector space, and the corresponding notion for a derived category is the finest such decomposition, which comes from exceptional objects.

Remark 2.2. Semiorthogonal decompositions are frequently defined by a different condition, namely that every object $T \in \mathcal{D}$ should admit a filtration

$$0 = T_{r+1} \to T_r \to \cdots \to T_1 \to T_0 = T \qquad \text{with} \quad \operatorname{cone}(T_{i+1} \to T_i) \in \mathcal{B}_i.$$

Semiorthogonality forces such a filtration to be unique up to isomorphism and functorial in T, and the objects admitting one form a thick subcategory. When each $\mathcal{B}_i$ is admissible, this condition is equivalent to the generation condition in the definition above (see [Kuz14, §2]). In particular, a semiorthogonal decomposition comes with projection functors $\mathcal{D} \to \mathcal{B}_i$ sending T to $\operatorname{cone}(T_{i+1} \to T_i)$; these are what make semiorthogonal decompositions usable in practice.

Definition 2.3. An object $E \in D(X)$ is exceptional if

$$\operatorname{Hom}_{D(X)}(E, E[\ell]) = \begin{cases} \mathbf{k} & \ell = 0, \\ 0 & \ell \neq 0. \end{cases}$$

Since $\mathrm{Hom}_{D(X)}(E, E[\ell]) = \mathrm{Ext}^\ell_X(E, E)$, this is equivalent to saying that $\mathrm{Hom}_X(E, E) = \mathbf{k}$ and all higher Ext groups vanish. An ordered sequence $E_0, \dots, E_r$ of exceptional objects is an exceptional collection if $\mathrm{Ext}^\bullet_X(E_i, E_j) = 0$ whenever $i > j$. It is strong if in addition $\mathrm{Ext}^\ell_X(E_i, E_j) = 0$ for all $i \leq j$ and all $\ell \neq 0$ (for a collection of sheaves the negative Ext groups vanish automatically, so only $\ell > 0$ needs to be checked), and full if the collection generates $\mathcal{D}$. We will abbreviate "full strong exceptional collection" to "FSEC" throughout.

Example 2.4. Let X be any projective toric variety (we will discuss background on toric varieties in Lectures 4 and 5). Any line bundle L on X is exceptional. This is because:

$$\mathrm{Hom}_X(L, L) = H^0(X, \mathcal{O}_X) = \mathbf{k},$$

since X is projective and integral. And we have

$$\mathrm{Ext}^\ell_X(L, L) = H^\ell(X, \mathcal{O}_X) = 0 \qquad \text{for all } \ell > 0,$$

by Demazure Vanishing (see Theorem 6.8 below) applied to the trivial divisor.

A full exceptional collection $E_0, \dots, E_r$ yields a semiorthogonal decomposition $\mathcal{D} = \langle E_0, \dots, E_r\rangle$ in which each block is the subcategory generated by a single E_i, itself equivalent to $D(\mathbf{k})$, the category of bounded complexes of $\mathbf{k}$-vector spaces. The fundamental example is due to Beilinson.

Theorem 2.5 ([Bei78])**.** *The projective space $\mathbb{P}^n$ admits two full strong exceptional collections:*

$$D(\mathbb{P}^n) = \langle \mathcal{O}(-n), \mathcal{O}(-n+1), \dots, \mathcal{O}\rangle = \langle \Omega^n(n), \dots, \Omega^1(1), \mathcal{O}\rangle.$$

Sketch of proof. We only address the first collection. We have $\mathrm{Ext}^\ell_{\mathbb{P}^n}(\mathcal{O}(i), \mathcal{O}(j)) = H^\ell(\mathbb{P}^n, \mathcal{O}(j-i))$. Recall the cohomology of line bundles on $\mathbb{P}^n$: $H^\ell(\mathbb{P}^n, \mathcal{O}(d)) = 0$ for $0 < \ell < n$ and every d, while $H^0(\mathbb{P}^n, \mathcal{O}(d)) = 0$ for $d < 0$ and $H^n(\mathbb{P}^n, \mathcal{O}(d)) = 0$ for $d > -(n+1)$. When $i \leq j$ the twist $j - i$ is nonnegative, so only H^0 survives; this gives the exceptional condition. When $i > j$ the twist lies in $\{-n, \dots, -1\}$, so all cohomology vanishes; this gives the strong condition. For fullness: given a coherent sheaf $\mathcal{F}$ on $\mathbb{P}^n$, Proposition 1.8 implies that $\mathcal{F} \cong \Phi_{\mathcal{O}_\Delta}(\mathcal{F})$ in $D(\mathbb{P}^n)$. We will see in Construction 10.5 that $\Phi_{\mathcal{O}_\Delta}(\mathcal{F})$ is isomorphic to a complex $\mathcal{B}$ with terms

$$\mathcal{B}_k = \bigoplus_{i-j=k} H^j(\mathbb{P}^n, \mathcal{F} \otimes \Omega^i(i)) \otimes \mathcal{O}(-i).$$

It follows that the objects $\mathcal{O}(-n), \mathcal{O}(-n+1), \dots, \mathcal{O}$ generate $D(\mathbb{P}^n)$. $\square$

Corollary 2.6. *For every sheaf $\mathcal{F}$ on $\mathbb{P}^n$, there is a (unique up to homotopy) complex F that is quasisomorphic to $\mathcal{F}$ and where $F_i = \bigoplus_{j=0}^n \mathcal{O}(-j)^{m_{ij}}$.*

Sketch of proof. Let K denote the homotopy category of complexes of graded S-modules whose terms are direct sums of line bundles in the Beilinson collection. There is a functor $K \to D^b(\mathbb{P}^n)$ that sends $S(i)$ to $\mathcal{O}(i)$ for $-n \leq i \leq 0$. Fully faithfulness (resp. essential surjectivity) of this functor follows from the strong exceptionality (resp. fullness) of the Beilinson collection. $\square$

Remark 2.7. Corollary 2.6 demonstrates what we mean about an "elegant global description". Any object in the derived category of coherent sheaves on $\mathbb{P}^n$ can be identified with a minimal free complex $F_\bullet$ where each F_i is a direct sum of $S(-n), S(-n+1), \cdots, S$, and that identification is unique up to isomorphism. So while $D(\mathbb{P}^n)$ may seem like a highly technical category, the corollary shows that it is in fact equivalent to something very concrete.

Theorem 2.5 and Corollary 2.6 provide a model for what we want from a global description of a derived category. By finding an FSEC of line bundles for $D(\mathbb{P}^n)$, we have broken $D(\mathbb{P}^n)$ into individual pieces, the $\mathcal{O}(-i)$, each of which we totally understand; this allows us to provide essentially canonical representations for any sheaf on $\mathbb{P}^n$, or more generally for any object in $D(\mathbb{P}^n)$, in terms of the very simple objects $\mathcal{O}(-n), \ldots, \mathcal{O}$ or, for the more algebraically minded, in terms of $S(-n), \ldots, S$.

A common metaphor is that finding an FSEC for a derived category is like finding a basis for a vector space. From an FSEC, one gets essentially unique representations of all objects as in Corollary 2.6, just like how a basis provides unique representations of vectors. The metaphor works on many levels: bases are non-unique, and FSEC's can be as well. Some bases are better than other: for instance, the monomials provide an especially nice basis for the vector space of degree d polynomials in n variables. Similarly, an FSEC consisting of simple objects like line bundles is especially desirable.

Another application of finding an FSEC is that we get an especially nice way to think about the Grothendieck group. Recall that the Grothendieck group $K_0(X)$ of a smooth projective variety X is the abelian group generated by the classes of coherent sheaves modulo the relations $[\mathcal{F}] = [\mathcal{E}] + [\mathcal{G}]$ for every short exact sequence $0 \to \mathcal{E} \to \mathcal{F} \to \mathcal{G} \to 0$. The existence of a full exceptional collection has a striking consequence for this group.

Corollary 2.8. *The Grothendieck group of $\mathbb{P}^n$ is $K_0(\mathbb{P}^n) \cong \mathbb{Z}^{n+1}$ with basis $[\mathcal{O}(-n)], [\mathcal{O}(-n+1)], \ldots, [\mathcal{O}]$.*

Proof. The classes of the objects in a full exceptional collection generate K_0, since the collection generates $D(\mathbb{P}^n)$. Semiorthogonality makes the matrix of the Euler pairing

$$\chi(E_i, E_j) = \sum_{\ell} (-1)^{\ell} \dim_{\mathbf{k}} \operatorname{Ext}^{\ell}_X(E_i, E_j)$$

upper triangular with 1's on the diagonal, hence invertible over $\mathbb{Z}$. Therefore the classes are linearly independent. $\square$

Remark 2.9. As an immediate corollary, for every sheaf $\mathcal{F}$ on $\mathbb{P}^n$, the class $[\mathcal{F}]$ in $K_0(\mathbb{P}^n)$ can be uniquely written as a combination of the classes $[\mathcal{O}(-n)], [\mathcal{O}(-n+1)], \ldots, [\mathcal{O}]$. Corollary 2.6 can be viewed as a categorification of this fact about $K_0(\mathbb{P}^n)$.

Inspired by Theorem 2.5, the past several decades have seen extensive searches for exceptional collections and semiorthogonal decompositions on other varieties. Beyond projective space, Kapranov constructed full exceptional collections on smooth quadrics, Grassmannians, and flag varieties [Kap88], while Orlov's blowup formula [Huy06, Prop. 11.18] and projective bundle formula [Huy06, Cor. 8.36] produce semiorthogonal decompositions from geometric constructions. These papers are just the tip of a giant iceberg; a full list of all papers involving the search for exceptional collections would involve hundreds of references.

Within the toric world, the search for exceptional collections led to King's conjecture in [Kin97], which proposed that something like Theorem 2.5 held for any smooth, projective toric variety. More specifically, King's Conjecture proposed that any such toric variety admitted an FSEC of line bundles. The conjecture received a great deal of attention for a period, though it turned out to be false. We will return to, and prove a revised version of, King's Conjecture in later lectures.

3. Lecture 3 (Isidora Bailly-Hall): Derived Category of $\mathbb{P}^n$, III

The previous lectures described $D(\mathbb{P}^n)$ through exceptional collections of sheaves, allowing for a concrete global description in terms of simple objects. In Lecture 18, we will use the theory of Tate resolutions to give a totally different, but equally concrete global description of $D(\mathbb{P}^n)$. This is a complementary and highly computational perspective, and it is built upon the Bernstein–Gel'fand–Gel'fand (BGG) correspondence [BGG78], which compares complexes of modules over a polynomial ring with complexes over its Koszul-dual exterior algebra. We follow [EFS03, §2] and return to this topic again when we introduce Tate resolutions in Lecture 18.

Let W be an $(n+1)$-dimensional $\mathbf{k}$-vector space with dual $V = W^*$, and set

$$S = \mathrm{Sym}(W) = \mathbf{k}[x_0, \ldots, x_n], \qquad E = \bigwedge(V) = \bigwedge(e_0, \ldots, e_n),$$

where $\{e_j\}$ is the basis of V dual to $\{x_j\}$, so that E is the exterior algebra on V, a finite-dimensional graded-commutative algebra in which $e_ie_j = -e_je_i$ and $e_j^2 = 0$. We grade these algebras so that $\deg(x_j) = 1$ and $\deg(e_j) = -1$, and $\mathbb{P}^n = \mathrm{Proj}\, S = \mathbb{P}(W)$. The two algebras are Koszul dual, meaning that there are $\mathbf{k}$-algebra isomorphisms $\mathrm{Ext}^*_S(\mathbf{k}, \mathbf{k}) \cong E$, and $\mathrm{Ext}^*_E(\mathbf{k}, \mathbf{k}) \cong S$. There are many other examples of Koszul dual algebras: see e.g. [PP05] for a survey.

3.1. **The BGG correspondence.** The correspondence is realized by an equivalence $R\colon D(S) \xrightarrow{\simeq} D(E)$ of bounded derived categories of graded modules: let us explain how this equivalence works. Let E^* denote the $\mathbf{k}$-dual of E, which is isomorphic as an E-module to $E(-n-1)$. For a graded S-module M, let $R(M)$ be the complex of free graded E-modules with

$$R(M)^i = E^*(-i) \otimes_{\mathbf{k}} M_i \quad \text{in cohomological degree } i,$$

with differential given by multiplication by $\sum_{j=0}^n e_j \otimes x_j$.

One extends R from modules to complexes by applying the above formula term-wise and totalizing. The inverse of R, which we denote by L, is given as follows. For a graded E-module P, let $L(P)$ be the complex of free graded S-modules with

$$L(P)^i = S(i) \otimes_{\mathbf{k}} P_{-i} \quad \text{in cohomological degree } i,$$

whose differential is also multiplication by $\sum_{j=0}^n x_j \otimes e_j$. One extends L to complexes in the same way as R.

Example 3.1. Say $S = \mathbf{k}[x_0, x_1]$ and $M = S/(x_0)$. Since $\dim_{\mathbf{k}} M_i = 1$ for $i \geq 0$, each nonzero term of $R(M)$ has rank 1. And since x_0 annihilates M, the differential is just multiplication by e_1:

$$R(M) = \quad \cdots \xleftarrow{e_1} E^*(-2) \xleftarrow{e_1} E^*(-1) \xleftarrow{e_1} E^*.$$

Example 3.2. Let S be as in Example 3.1, but let us now compute $R(S)$. Since $\dim_{\mathbf{k}} S_i = i+1$ for $i \geq 0$, we have $\mathrm{rank}\, R(S)^i = i+1$ for $i \geq 0$: notice how the Hilbert function of S gets encoded as the Betti numbers of $R(S)$. The complex $R(S)$ looks as follows:

$$R(S) = \quad \cdots \leftarrow E^*(-2)^3 \xleftarrow{\begin{pmatrix} e_0 & 0 \\ e_1 & e_0 \\ 0 & e_1 \end{pmatrix}} E^*(-1)^2 \xleftarrow{\begin{pmatrix} e_0 \\ e_1 \end{pmatrix}} E^*.$$

Example 3.3. Since $\mathbf{k}$ is concentrated in internal degree 0, the complex $R(\mathbf{k})$ has a single nonzero term, the free module E^* in cohomological degree 0. Now to apply L, note that the graded piece $(E^*)_p = \bigwedge^p W$ contributes the term $S(-p) \otimes_{\mathbf{k}} \bigwedge^p W$ in cohomological degree $-p$, and e_j acts on E^* by contraction, so the differential coming from $\sum_{i=0}^n x_i \otimes e_i$ assembles these terms into the Koszul complex

$$L(E^*)\colon \quad S \leftarrow S \otimes \bigwedge^1 W \leftarrow \cdots \leftarrow S \otimes \bigwedge^{n+1} W,$$

sitting in cohomological degrees $-n-1, \ldots, 0$, which is the minimal free resolution of $\mathbf{k}$ over S. Hence $LR(\mathbf{k}) \simeq \mathbf{k}$ in $D(S)$, as Theorem 3.5 predicts. The role of the twist in $E^* \cong E(-n-1)$ is that applying L to the untwisted module E instead yields the dual Koszul complex involving the $\bigwedge^p V$, whose only cohomology is $\mathbf{k}(n+1)$ in cohomological degree $n+1$.

Of course, the Koszul complex, which is a free resolution of $\mathbf{k}$ over S, can be viewed as an exterior algebra. So one can construct a natural isomorphism $\mathbf{k} \cong L(E^*) \cong S \otimes_{\mathbf{k}} E$ in $D(S)$ (ignoring grading twists). Similarly, we have $\mathbf{k} \cong RL(\mathbf{k}) = R(S) = S \otimes_{\mathbf{k}} E$ in $D(E)$ (once again ignoring gradings).

That is, the injective resolution of $\mathbf{k}$ over E is naturally isomorphic to a symmetric algebra over E! This is a key idea behind Koszul duality; see e.g. [?PP05] for more.

Notice that $\operatorname{rank}_E R(M) = \dim_{\mathbf{k}} M$, and similarly for L. Thus, applying L to any finitely generated E-module gives a bounded complex. But since finitely generated S-modules are rarely finitely dimensional over $\mathbf{k}$, we typically have $R(M)_i \neq 0$ for $i \ll 0$. Nevertheless, if M is finitely generated, the homology of $R(M)$ is as well, by the following lemma:

Lemma 3.4 ([?EFS, Proposition 2.3(a)]). *For any graded S-module M, we have $H_i(R(M))_j \cong \operatorname{Tor}^S_{i+j}(M, \mathbf{k})_j$.*

In particular, the functor R sends finitely generated S-modules to the *bounded* derived category of E. We include a sketch of a proof, since we will need this lemma again in Lecture 18.

Sketch of proof. The idea is that, forgetting gradings, $R(M)$ is isomorphic to $M \otimes_S K$, where K is the Koszul complex on the variables in S. The proof thus boils down to keeping careful track of how this isomorphism interacts with the cohomological and internal gradings; we omit the details. $\square$

We omit the proof of the following theorem; a complete argument is more than one lecture allows.

Theorem 3.5 ([?EFS, Corollary 2.7], [?BGG]). *The functors L and R descend to mutually inverse equivalences of triangulated categories*

$$R\colon D(S) \xrightarrow{\simeq} D(E), \qquad L\colon D(E) \xrightarrow{\simeq} D(S).$$

Let us illustrate the equivalence by evaluating the composition at the residue field $\mathbf{k} = S/\mathfrak{m}$, where $\mathfrak{m} = (x_0, \ldots, x_n)$.

Remark 3.6. For a finitely generated graded S-module M, the complex $\mathbf{R}(M)$ is exact in cohomological degrees $i > \operatorname{reg} M$, and conversely the Castelnuovo–Mumford regularity of M can be detected on the exterior side by applying $\mathbf{R}$ to the truncations $M_{\geq d}$ [?EFS, §2]. Regularity is usually defined through the vanishing of graded Betti numbers over S; under the BGG correspondence it becomes literal cohomological vanishing over E. This reformulation powers the Tate resolutions of Lecture 18.

3.2. **The singularity category of E.** The equivalence of Theorem 3.5 interacts with two natural subcategories, and passing to quotients recovers $D(\mathbb{P}^n)$. Let $D_{\mathfrak{m}}(S) \subseteq D(S)$ denote the full subcategory given by complexes whose homology is finite dimensional over $\mathbf{k}$, and let $\operatorname{Perf}(E)$ be the full subcategory of *perfect complexes*, i.e. objects that are quasi-isomorphic to a bounded complex of finitely generated free modules. Since $D_{\mathfrak{m}}(S)$ (resp. $\operatorname{Perf}(E)$) is the thick subcategory generated by twists of $\mathbf{k}$ (resp. E), and the functor R maps twists of $\mathbf{k}$ to twists (and cohomological shifts) of E, one concludes:

Proposition 3.7 ([EFS03, §2]). *Under the equivalence of Theorem 3.5, the subcategory $D_{\mathfrak{m}}(S)$ corresponds to* $\operatorname{Perf}(E)$.

Remark 3.8. Proposition 3.7 exchanges two finiteness conditions that live on opposite sides of the correspondence. Over S, which has finite global dimension, every object of $D(S)$ is a perfect complex, whereas having finite-length homology is a strong restriction. Over E the situation is reversed: every object of $D(E)$ has finite-dimensional homology, since E itself is finite-dimensional, whereas perfection is a strong restriction because E has infinite global dimension. For instance, the residue field $\mathbf{k}$ of E has an infinite minimal free resolution. This asymmetry explains why the same Verdier quotient construction below produces a geometric category on the symmetric side and a singularity category on the exterior side.

Definition 3.9 (see [Wei94, Ch. 10])**.** Given a thick subcategory $\mathcal{B} \subseteq \mathcal{D}$, the Verdier quotient $\mathcal{D}/\mathcal{B}$ is the triangulated category with the same objects as $\mathcal{D}$, obtained by formally inverting all morphisms whose cone lies in $\mathcal{B}$.

Applying this to $D_{\mathfrak{m}}(S)$ recovers the geometry of $\mathbb{P}^n$: there is an equivalence

$$D(S) \,/\, D_{\mathfrak{m}}(S) \simeq D(\mathbb{P}^n),$$

which is the derived-category analogue of Serre's description of coherent sheaves on $\operatorname{Proj} S$ as finitely generated graded S-modules modulo those of finite length (compare Lecture 5, where the analogous statement $D(X) \simeq D(S)/D_B(S)$ is discussed for toric X).

On the exterior side, the analogous quotient is the singularity category

$$D_{sg}(E) \coloneqq D(E) \,/\, \operatorname{Perf}(E),$$

in the sense of Buchweitz and Orlov [Buc21, Orl09a]; since E is self-injective, $\mathrm{D}_{\mathrm{sg}}(E)$ is equivalent to the stable module category of E. Combining Theorem 3.5 with Proposition 3.7 and passing to Verdier quotients, we conclude:

$$D(\mathbb{P}^n) \simeq D(S)/D_{\mathfrak{m}}(S) \simeq D(E)/\operatorname{Perf}(E) = D_{sg}(E).$$

That is, the BGG correspondence identifies the derived category of $\mathbb{P}^n$ with the singularity category of the exterior algebra E. We will return to this equivalence in Lecture 18 and discuss its generalization to toric varieties in Lecture 19.

4. Lecture 4 (Sharon Robbins): Toric varieties and Cox rings, I

We now turn from projective space to the wider world of toric varieties, whose combinatorial description makes them an ideal candidate for scrutinizing the structure of derived categories. Throughout, $N \cong \mathbb{Z}^d$ is the lattice of one-parameter subgroups of a torus $T = N \otimes_{\mathbb{Z}} \mathbf{k}^* \cong (\mathbf{k}^*)^d$ and its dual $M = \mathrm{Hom}_{\mathbb{Z}}(N, \mathbb{Z})$ is the character lattice of T. The usual, somewhat cryptic incantation is: a toric variety is a normal variety X containing T as a dense open subset, together with an action of T on X extending its action on itself. While this is an accurate definition of a toric variety, what makes it cryptic is that it does not indicate in any way why one should care about toric varieties.

There are a few reasons to care about toric varieties. The first is simply that they show up in lots of different areas of math. This is not especially surprising, as many common varieties are toric varieties, including projective space, products of projective spaces, weighted projective spaces, Hirzebruch surfaces, and much more. The second reason to care about toric varieties is that they are a rich enough class of varieties to exhibit genuinely subtle behavior, but they are a concrete enough class of varieties to allow for computational and experimental methods of study. A third reason is that toric varieties have close connections to combinatorics, and thus often show up in relation to questions at the boundary of algebraic geometry and combinatorics.

Rather than pursuing these connections, the next two lectures focus on the combinatorial and algebraic structure of toric varieties. Our references are [CLS11, Ch. 3–5] and David Cox's paper [Cox95].

4.1. **Toric varieties from fans.** An affine toric variety is built from a cone. Let $\sigma \subseteq N \otimes \mathbf{R}$ be a strongly convex rational polyhedral cone, with dual cone $\sigma^\vee \subseteq M \otimes \mathbf{R}$. Then $\sigma^\vee \cap M$ is a finitely generated semigroup, and $U_\sigma = \operatorname{Spec} \mathbf{k}[\sigma^\vee \cap M]$ is an affine toric variety. Using these affine pieces, an arbitrary normal toric variety is glued together along the combinatorics of a fan.

Definition 4.1. A fan Σ in $N \otimes \mathbf{R}$ is a finite collection of strongly convex rational polyhedral cones such that (i) every face of a cone in Σ is again in Σ, and (ii) the intersection of any two cones in Σ is a face of each. The associated toric variety X_Σ is obtained by gluing the affine charts $\{U_\sigma\}_{\sigma \in \Sigma}$ along the charts of their common faces.

The dictionary between the geometry of X_Σ and combinatorics of Σ is explained in [CLS11, §3.1]. In particular X_Σ is smooth (resp. simplicial) if every cone of Σ is generated by part of a $\mathbb{Z}$-basis (resp. an $\mathbf{R}$-basis) of N, and it is complete (equivalently proper) exactly when Σ covers all of $N \otimes \mathbf{R}$. We assume from now on that the rays of Σ span $N \otimes \mathbf{R}$, i.e. that X_Σ has no torus factor; this is automatic when X_Σ is complete, as all of our examples are.

Example 4.2. One of the simplest fans is the fan for $\mathbb{P}^1$. In this case, $N \cong \mathbb{Z}^1$ and $\Sigma \subseteq N \otimes \mathbf{R}$ consists of 3 cones: the ray $\mathbf{R}_{\geq 0}$; the ray $\mathbf{R}_{\leq 0}$; and the origin $\{0\}$, which is a face of each of the other two.

When $\sigma = \mathbf{R}_{\geq 0}$ then $\sigma^\vee = \mathbf{R}_{\geq 0}$ and $\sigma^\vee \cap \mathbb{Z} = \mathbb{Z}_{\geq 0}$. Thus the semigroup ring $\mathbf{k}[\sigma^\vee \cap \mathbb{Z}] = \mathbf{k}[\mathbb{Z}_{\geq 0}] = \mathbf{k}[t]$. Similarly, the cone $\mathbf{R}_{\leq 0}$ yields the semigroup ring $\mathbf{k}[t^{-1}]$. These are glued along the semigroup ring $\mathbf{k}[t^{-1}, t]$. Thus this fan corresponds to the projective line $\mathbb{P}^1$.

4.2. **Rays, divisors, and the class group.** We will write $\Sigma(1)$ for the set of rays (one-dimensional cones) of Σ, and $u_\rho \in N$ will denote the primitive generator of a ray ρ. By the Orbit–Cone Correspondence [CLS11, §3.2], each ray ρ corresponds to an irreducible T-invariant Weil divisor D_ρ on X_Σ, and every T-invariant divisor is an integral combination $\sum_\rho a_\rho D_\rho$. Identifying $\mathrm{Div}_T(X_\Sigma) = \mathbb{Z}^{\Sigma(1)}$, there is an exact sequence [CLS11, Thm. 4.1.3]

$$0 \to M \xrightarrow{\alpha} \mathbb{Z}^{\Sigma(1)} \xrightarrow{\deg} \mathrm{Cl}(X_\Sigma) \to 0, \qquad \alpha(m) = \sum_\rho \langle m, -u_\rho \rangle D_\rho, \tag{4.3}$$

where $\mathrm{Cl}(X_\Sigma)$ is the divisor class group. The map α gives the principal divisor of the character χ^m, and deg sends a divisor to its class; injectivity of α uses our assumption that the rays span $N \otimes \mathbf{R}$.

Remark 4.4 (Cartier and Weil divisors). Inside the class group sits the Picard group $\mathrm{Pic}(X_\Sigma) \hookrightarrow \mathrm{Cl}(X_\Sigma)$ of *Cartier* divisor classes. The toric variety is smooth if and only if the inclusion $\mathrm{Pic}(X_\Sigma) \subseteq \mathrm{Cl}(X_\Sigma)$ is an equality (an abstract isomorphism of the two groups is not enough: on $\mathbb{P}(1,1,2)$ one has $\mathrm{Pic} = 2\mathbb{Z} \subsetneq \mathbb{Z} = \mathrm{Cl}$, and $2\mathbb{Z} \cong \mathbb{Z}$), and it is simplicial if and only if $\mathrm{Pic}(X_\Sigma)$ has finite index in $\mathrm{Cl}(X_\Sigma)$, in which case every Weil divisor is $\mathbb{Q}$-Cartier, meaning some positive multiple is Cartier [CLS11, Prop. 4.2.6, 4.2.7]. Example 4.8 below illustrates both phenomena on $\mathbb{P}(1,1,2)$; the distinction is also apparent in the module–sheaf correspondence and in Demazure vanishing in Lecture 5.

4.3. **The Cox ring and irrelevant ideal.** Assign to each ray $\rho \in \Sigma(1)$ a variable x_ρ and consider

$$S = \mathrm{Cox}(X_\Sigma) = \mathbf{k}[x_\rho : \rho \in \Sigma(1)]$$

to be a polynomial ring graded by $\mathrm{Cl}(X_\Sigma)$ via $\deg(x_\rho) = [D_\rho]$. This is the Cox ring (or homogeneous coordinate ring) of X_Σ. A monomial $x^{\mathbf{a}} = \prod_\rho x_\rho^{a_\rho}$ has degree $[\sum_\rho a_\rho D_\rho] \in \mathrm{Cl}(X_\Sigma)$.

The geometry of X_Σ, encoded in the combinatorics of the maximal cones $\Sigma_{\max}$ of Σ, is recorded by the so-called irrelevant ideal

$$B = \big\langle\, x_{\widehat{\sigma}} \;:\; \sigma \in \Sigma_{\max} \,\big\rangle, \qquad x_{\widehat{\sigma}} = \prod_{\rho \,\notin\, \sigma(1)} x_\rho,$$

which is a squarefree monomial ideal generated, for each maximal cone σ, by the product of the variables whose rays are not in σ.

Remark 4.5 (Primitive collections). The locus $V(B) \subseteq \mathbb{A}^{\Sigma(1)}$ cut out by the irrelevant ideal is a union of coordinate subspaces, indexed by the primitive collections of Σ: the minimal subsets $\mathcal{C} \subseteq \Sigma(1)$ that do not span a cone of Σ, although every proper subset does. Explicitly,

$$V(B) = \bigcup_{\mathcal{C}} V(x_\rho : \rho \in \mathcal{C}) \quad \text{[CLS11, Prop. 5.1.6].}$$

Observe that the ring S, its grading, and the group G below depend only on the rays $\Sigma(1)$, whereas B and $V(B)$ remember the maximal cones; in particular, different fans with the same rays share the Cox ring S but not the ideal B. We return to this observation in Lecture 5 and, systematically, in Lecture 11.

Theorem 4.6 (quotient construction, [CLS11, Thm. 5.1.11])**.** *Let X_Σ be a toric variety with no torus factor and let $G = \mathrm{Hom}_{\mathbb{Z}}(\mathrm{Cl}(X_\Sigma), \mathbf{k}^*)$ act on* $\mathrm{Spec}\, S = \mathbb{A}^{\Sigma(1)}$ *via the grading on S. Then*

$$X_\Sigma \cong \big(\mathrm{Spec}\, S \setminus V(B)\big) /\!\!/\, G$$

is a good categorical quotient, and it is a geometric quotient if and only if X_Σ is simplicial.

Thus a point of a simplicial X_Σ is a G-orbit of "homogeneous coordinates" $[a_\rho]_{\rho\in\Sigma(1)}$ not lying in the excluded locus $V(B)$. For $\mathbb{P}^n$ this recovers the familiar $\mathbb{P}^n = (\mathbb{A}^{n+1} \setminus \{0\}) \,/\, \mathbf{k}^*$.

Example 4.7. We now illustrate several toric varieties that we will use as running examples.

(1) **Projective space $\mathbb{P}^n$.** The fan of $\mathbb{P}^n$ has as rays the standard basis vectors $u_1, \ldots, u_n$ as well as $u_0 = -u_1 - \cdots - u_n$, with maximal cones spanned by all n-subsets; see Figure 4.1. The Cox ring S is $\mathbf{k}[x_0, \ldots, x_n]$ with the standard $\mathbb{Z}$-grading $\deg(x_i) = 1$ and irrelevant ideal $B = (x_0, \ldots, x_n)$.

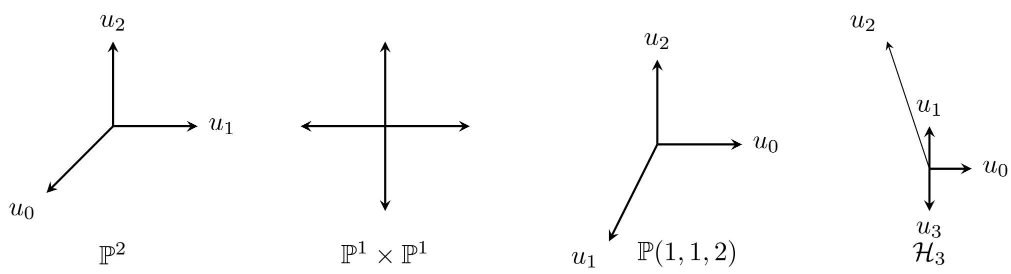


FIGURE 4.1. Fans of the two-dimensional running examples.

(2) **Products $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_r}$.** The fan is the product of the fans of the factors. For instance, in the case of $\mathbb{P}^1 \times \mathbb{P}^1$, the rays are $(\pm 1, 0)$ and $(0, \pm 1)$, as depicted in Figure 4.1. The Cox ring is $\mathbf{k}[x_{1,0}, \ldots, x_{1,n_1}, \ldots, x_{r,0}, \ldots, x_{r,n_r}]$, with $\mathrm{Cl}(X) = \mathbb{Z}^r$-grading such that $\deg(x_{i,j})$ is the i^{th} standard basis vector of $\mathbb{Z}^r$. The irrelevant ideal is $B = \bigcap_i (x_{i,0}, \ldots, x_{i,n_i})$, the intersection of the irrelevant ideals of each factor $\mathbb{P}^{n_i}$.

(3) **Weighted projective space $\mathbb{P}(1,1,2)$.** The rays of the fan are $u_0 = (1,0)$, $u_1 = (-1,-2)$, and $u_2 = (0,1)$, and the maximal cones, just as with $\mathbb{P}^2$, are those spanned by any pair of rays; see Figure 4.1. Since $u_0 + u_1 + 2u_2 = 0$, the Cox ring of $\mathbb{P}(1,1,2)$ is $\mathbf{k}[x_0, x_1, x_2]$, with $\mathbb{Z}$-grading given by $\deg(x_0) = \deg(x_1) = 1$, and $\deg(x_2) = 2$. Just as in the case of $\mathbb{P}^2$, the irrelevant ideal is $B = (x_0, x_1, x_2)$. Note that the inclusion of the Picard group in the class group is $2\mathbb{Z} \hookrightarrow \mathbb{Z}$, and so $\mathbb{P}(1,1,2)$ is singular; see Example 4.8. This example generalizes to any weighted projective space $\mathbb{P}(a_0, \ldots, a_n)$ with $\gcd(a_0, \ldots, a_n) = 1$; if moreover every n-element subset of $\{a_0, \ldots, a_n\}$ has gcd 1, the presentation is called well formed, and the weights are then recovered from the variety.

(4) **Hirzebruch surfaces $\mathcal{H}_a$ ($a \geq 1$).** The surface $\mathcal{H}_a = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(a))$ is the toric variety of the fan in $N = \mathbb{Z}^2$ with rays

$$u_0 = e_1, \qquad u_1 = e_2, \qquad u_2 = -e_1 + a e_2, \qquad u_3 = -e_2,$$

and the four maximal cones spanned by neighboring pairs (Figure 4.1). Writing x_i for the variable of u_i, the sequence (4.3) gives the relations $D_0 \sim D_2$ and $D_3 \sim D_1 + aD_2$, so in the basis $([D_2], [D_3])$ of $\mathrm{Cl}(\mathcal{H}_a) = \mathbb{Z}^2$ the degrees are

$$\deg(x_0) = \deg(x_2) = (1,0), \qquad \deg(x_1) = (-a, 1), \qquad \deg(x_3) = (0,1),$$

and the irrelevant ideal is $B = (x_0, x_2) \cap (x_1, x_3)$. Geometrically, $(1,0)$ is the class of a fiber of $\mathcal{H}_a \to \mathbb{P}^1$, and $(-a, 1)$ is the class of the divisor with self-intersection $-a$.

Example 4.8 ($\mathbb{P}(1,1,2)$ in detail)**.** Let $S = \mathbf{k}[x,y,z]$ with $\deg(x) = \deg(y) = 1$ and $\deg(z) = 2$ and $B = (x,y,z)$. By Theorem 4.6, $\mathbb{P}(1,1,2) = (\mathbb{A}^3 \setminus \{0\}) \,/\, \mathbf{k}^*$ for the action $\lambda \cdot (x,y,z) = (\lambda x, \lambda y, \lambda^2 z)$. The irrelevant ideal gives the open affine cover $D_+(x) \cup D_+(y) \cup D_+(z)$, where

$$D_+(x) = \mathrm{Spec}\big(S[x^{-1}]\big)_0 = \mathrm{Spec}\,\mathbf{k}\left[\tfrac{y}{x}, \tfrac{z}{x^2}\right] \cong \mathbb{A}^2,$$

and analogously for $D_+(y) \cong \mathbb{A}^2$, but

$$D_+(z) = \mathrm{Spec}\big(S[z^{-1}]\big)_0 = \mathrm{Spec}\,\mathbf{k}\left[\tfrac{x^2}{z}, \tfrac{xy}{z}, \tfrac{y^2}{z}\right] \cong \mathrm{Spec}\,\mathbf{k}[u,v,w]/(uw - v^2)$$

is the quadric cone. Thus $\mathbb{P}(1,1,2)$ is singular exactly at the point $[0:0:1]$, the image of the locus $\{x = y = 0\}$ on which the $\mathbf{k}^*$-action has stabilizer $\mu_2 = \{\pm 1\}$. In particular, a divisor class $d \in \mathrm{Cl}(\mathbb{P}(1,1,2)) = \mathbb{Z}$ is Cartier if and only if d is a multiple of $2 = \mathrm{lcm}(1,1,2)$, so $\mathrm{Pic}(\mathbb{P}(1,1,2)) = 2\mathbb{Z} \subsetneq \mathbb{Z}$, as Remark 4.4 predicts for a simplicial but non-smooth toric variety. In particular the rank-one reflexive sheaf $\mathcal{O}(1)$, corresponding to the module $S(1)$, is *not* a line bundle. This point is further scrutinized in Lecture 5.

5. Lecture 5 (Caitlin Davis): Toric varieties and Cox rings, II

Continuing from Lecture 4, we describe the dictionary between the commutative algebra of the Cox ring S and the geometry of a simplicial toric variety $X = X_\Sigma$, culminating in a derived equivalence that generalizes the description of $D(\mathbb{P}^n)$ from Lecture 3.

5.1. The ideal–variety correspondence.

Just as homogeneous ideals of $\mathbf{k}[x_0, \ldots, x_n]$ cut out closed subschemes of $\mathbb{P}^n$, homogeneous ideals of S cut out closed subschemes of X, provided we account for the irrelevant ideal B.

Proposition 5.1 (Toric Ideal–Variety Correspondence, [CLS11, Prop. 5.2.7])**.** *Let X be a simplicial toric variety with Cox ring S and irrelevant ideal B. There is a bijective correspondence*

$$\{\textit{closed subvarieties of } X\} \longleftrightarrow \{\textit{radical homogeneous ideals } I \subseteq B\}.$$

Note that the main result of [BEF26] extends this to the nonsimplicial case. Instead of intersecting with B as in Proposition 5.1, an alternate bijection arises if one replaces I by its B-saturation instead. Recall that the saturation of an ideal I with respect to an ideal J is denoted $I : J^\infty$ and is the ideal

$$I : J^\infty := \{f \in S : J^k f \subseteq I \text{ for some } k\}.$$

We will say that a homogeneous ideal $I \subseteq S$ is B-saturated if $I = I : B^\infty$. At the level of subschemes the correspondence requires smoothness:

Proposition 5.2 ([CLS11, Prop. 6.A.7])**.** *On a smooth toric variety X, two homogeneous ideals $I, J \subseteq S$ determine the same ideal sheaf (hence the same closed subscheme) if and only if $(I : B^\infty) = (J : B^\infty)$. Consequently, closed subschemes of X correspond bijectively to B-saturated homogeneous ideals of S.*

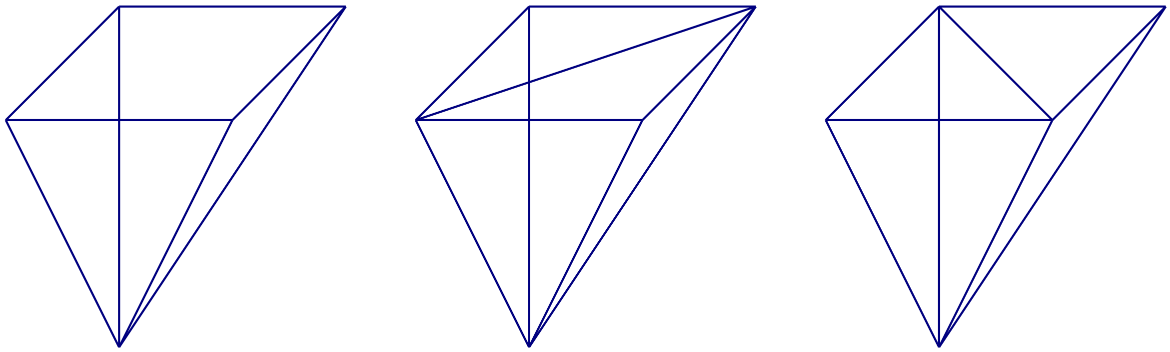
The analogue of this in the simplicial case appears in [Cox95, Thm. 3.7].

5.2. The Cox ring does not remember the variety.

Distinct toric varieties can have the same Cox ring. Let us return to the quotient construction of a toric variety from Theorem 4.6. In that case, the toric variety is realized as a quotient of $\mathrm{Spec}(S) \setminus V(B)$ by a particular group action. The data of the group action on $\mathrm{Spec}(S)$ is equivalent to the data of the multigrading of the Cox ring. But given this data, there can be distinct irrelevant ideals which yield distinct toric quotients.

At the level of fans, we can say it this way: as observed in Remark 4.5, the ring S, its grading, and the group G depend only on the rays of Σ, while the irrelevant ideal B remembers the maximal cones. Thus, if we have two complete fans with the same rays but different maximal cones, these will correspond to distinct toric varieties with the same Cox ring. This is a subtlety that is absent from projective space. In that case, there is only one complete fan with the given set of rays.

Example 5.3. To see an example of this phenomenon, we need to go to three dimensions. Imagine a 3-dimensional fan containing a cone like the one in picture (a) below.



(a) (b) (c)

There are two distinct ways to subdivide this into a pair of simplicial cones without adding any rays: the two ways correspond to adding one of the two diagonal planes, as illustrated in pictures (b) and (c) above.

Now, there are a number of ways that you could extend either (b) or (c) to a complete simplicial fan by adding the same rays and cones everywhere outside of this cone. If you do that, then you will end up with two distinct complete and simplicial fans with the same rays. The corresponding toric varieties will have the same Cox ring.

Example 5.4. A slightly different phenomenon occurs in our running example of the Hirzebruch surface. Let $S = \mathbf{k}[x_0, x_1, x_2, x_3]$ carry the $\mathbb{Z}^2$-grading of Example 4.7(4) with $a = 3$, so that

$$\deg(x_0) = \deg(x_2) = (1, 0), \qquad \deg(x_1) = (-3, 1), \qquad \deg(x_3) = (0, 1),$$

and let $G = (\mathbf{k}^*)^2$ act accordingly, i.e. $(\lambda, \mu) \in G$ acts by

$$(\lambda, \mu) \cdot x_i = \begin{cases} \lambda x_0 & i = 0 \\ \lambda^{-3} \mu x_1 & i = 1 \\ \lambda x_2 & i = 2 \\ \mu x_3 & i = 3. \end{cases}$$

Two different squarefree monomial ideals yield two very different geometric quotients:

- $B = (x_0, x_2) \cap (x_1, x_3)$ yields the Hirzebruch surface $\mathcal{H}_3$, which is smooth;
- $B' = (x_1) \cap (x_0, x_2, x_3)$ yields the weighted projective space $\mathbb{P}(1, 1, 3)$, which is singular. To see this, we split $G = (\mathbf{k}^*)^2$ as a product $\{(1, \mu)\} \times \{(s, s^3)\}$ of two copies of $\mathbf{k}^*$. On the locus $x_1 \neq 0$, we can use the first $\mathbf{k}^*$-factor to scale x_1 to 1. Then $\big(\mathrm{Spec}(S) \setminus V(x_1)\big)/\mathbf{k}^* \cong \big(\mathbb{A}^3 \times \mathbf{k}^*\big)/\mathbf{k}^* \cong \mathbb{A}^3$. The second $\mathbf{k}^*$-factor, $\mathbf{k}^* = \{(s, s^3)\} \subseteq G$, then acts trivially on x_1 and so it passes to an action on this quotient $\mathbb{A}^3$, acting on (x_0, x_2, x_3) with weights $(1, 1, 3)$. This exhibits the full quotient as
$$(\mathbb{A}^4 - V(B'))/G \cong (\mathbb{A}^3 \setminus \{0\}) \,/\, \mathbf{k}^* = \mathbb{P}(1, 1, 3).$$

Since the two quotients are not isomorphic, the Cox ring together with the group action does not determine the variety: the irrelevant ideal is essential extra data. This example is subtler than the previous one, however, because only the first of the two presentations is the Cox construction of Theorem 4.6: the Cox ring of $\mathbb{P}(1, 1, 3)$ itself is the *three*-variable ring $\mathbf{k}[y_0, y_1, y_2]$ with $\mathrm{Cl}(\mathbb{P}(1, 1, 3)) = \mathbb{Z}$ and degrees $(1, 1, 3)$. This is because the ray corresponding to x_1 does not appear in its fan; equivalently, the ideal (x_1) is a principal component of B'.

It is natural to wonder: given a multigraded Cox ring, or given a set of rays, what are all of the projective, simplicial toric varieties that can arise? This question leads to the study of the secondary fan, which parametrizes all the toric varieties arising from a fixed graded polynomial ring; see Example 11.3 in Lecture 11 for details on this.

5.3. **Modules to sheaves.** Every finitely generated graded S-module M gives rise to a coherent sheaf $\widetilde{M}$ on X, following [CLS11, §5.3]. On the affine chart U_σ associated to a maximal cone σ, the sections are the degree-zero part of the localization at the monomial $x^{\widehat{\sigma}} = \prod_{\rho \notin \sigma(1)} x_\rho$:

$$\Gamma\big(U_\sigma, \widetilde{M}\big) = \big(M[\,x^{-\widehat{\sigma}}\,]\big)_0.$$

These glue to a sheaf $\widetilde{M}$ on X, and $M \mapsto \widetilde{M}$ is exact. For a class $\mathbf{d} \in \mathrm{Cl}(X)$, the module $S(\mathbf{d})$ sheafifies to the rank-one reflexive sheaf $\mathcal{O}_X(\mathbf{d}) = \widetilde{S(\mathbf{d})}$.

Remark 5.5. When X is smooth, the sheaves $\mathcal{O}_X(\mathbf{d})$ are line bundles, $\mathcal{O}(\mathbf{d}) \otimes \mathcal{O}(\mathbf{e}) \cong \mathcal{O}(\mathbf{d}+\mathbf{e})$, and twisting commutes with sheafification: $\widetilde{M(\mathbf{d})} \cong \widetilde{M} \otimes \mathcal{O}(\mathbf{d})$. All three statements may fail for classes outside $\mathrm{Pic}(X)$ when X is merely simplicial: on $\mathbb{P}(1,1,2)$ the sheaf $\mathcal{O}(1)$ is not a line bundle (Example 4.8), and the natural map $\mathcal{O}(1) \otimes \mathcal{O}(1) \to \mathcal{O}(2)$ is not an isomorphism. Handling such twists correctly is one motivation for the toric stacks introduced below.

Remark 5.6 (Local cohomology). This remark is optional, but may be helpful for those familiar with local cohomology. To understand the difference between finitely generated graded S-modules and sheaves on X, it is helpful to recall some facts about local cohomology. For a $\mathrm{Cl}(X)$-graded S-module M, the B-torsion submodule

$$H^0_B(M) = \{\, m \in M : B^k m = 0 \text{ for some } k \geq 0 \,\}$$

is the largest submodule of M supported on $V(B)$, and $H^i_B(-)$ denotes the higher right derived functors of the left exact functor $H^0_B(-)$, computed from the Čech complex on a generating set of B [CLS11, §9]. There is a four-term exact sequence of $\mathrm{Cl}(X)$-graded modules

$$0 \longrightarrow H^0_B(M) \longrightarrow M \longrightarrow \bigoplus_{\mathbf{d} \in \mathrm{Cl}(X)} H^0\big(X, \widetilde{M(\mathbf{d})}\big) \longrightarrow H^1_B(M) \longrightarrow 0,$$

together with the isomorphisms $H^p\big(X, \widetilde{M(\mathbf{d})}\big) \cong H^{p+1}_B(M)_{\mathbf{d}}$ for $p \geq 1$. Thus $H^0_B(M)$ is exactly the part of M that sheafification forgets, while $H^1_B(M)$ measures the failure of M to already consist of all twisted global sections.

5.4. **Derived categories of toric varieties.** There is a functor that sends a finitely generated, graded S-module M to the corresponding coherent sheaf $\widetilde{M}$ on X. The kernel of this functor separates the geometry of X from the algebra of S. Write $\mathrm{mod}\,\mathrm{gr}(S)$ for the category of finitely generated $\mathrm{Cl}(X)$-graded S-modules, and $\mathrm{mod}_B(S) \subseteq \mathrm{mod}_{\mathrm{gr}}(S)$ for the full subcategory of B-torsion modules, i.e. those annihilated by a power of B.

Proposition 5.7 ([CLS11, Prop. 5.3.9 and 5.3.10])**.** *Let X be a toric variety with Cox ring S and irrelevant ideal B.*

1. *Every quasicoherent sheaf on X is isomorphic to $\widetilde{M}$ for some graded S-module M, which may be chosen finitely generated when the sheaf is coherent.*
2. *Let $M \in \mathrm{mod}_{\mathrm{gr}}(S)$. If M is B-torsion, i.e. if $M \in \mathrm{mod}_B(S)$, then $\widetilde{M} = 0$.*
3. *If X is* smooth *and $M \in \mathrm{mod}_{\mathrm{gr}}(S)$, then $\widetilde{M} = 0$ if and only if M is B-torsion.*

Consequently, sheafification $M \mapsto \widetilde{M}$ always induces a functor $\mathrm{mod}_{\mathrm{gr}}(S) / \mathrm{mod}_B(S) \to \mathrm{coh}(X)$, and when X is smooth, this yields an equivalence of abelian categories

$$\mathrm{coh}(X) \;\simeq\; \mathrm{mod}_{\mathrm{gr}}(S) / \mathrm{mod}_B(S), \tag{5.8}$$

which follows from Cox's work [Cox95, §3]. Passing to derived categories yields:

Theorem 5.9. *Let $X = X_\Sigma$ be a smooth toric variety with no torus factor, with Cox ring S and irrelevant ideal B, and let $D_B(S) \subseteq D(S)$ be the thick subcategory of complexes with homology supported on $V(B)$. Sheafification induces an equivalence of triangulated categories*

$$D(X) \;\simeq\; D(S)\,/\,D_B(S).$$

This generalizes the equivalence $D(\mathbb{P}^n) \simeq D(S)/D_{\mathfrak{m}}(S)$ from Lecture 3; it follows from the abelian equivalence in (5.8), and is also the smooth case of Corollary 5.13 below. Smoothness is essential:

Example 5.10 ([CLS11, Ex. 5.3.11])**.** On $\mathbb{P}(1,1,2)$, let $M = \big(S/(x,y)\big)(1)$. Note that M is not B-torsion: as $M \cong \mathbf{k}[z](1)$, its support in $\mathbb{A}^3$ is the entire z-axis, which is not contained in $V(B) = \{0\}$. Concretely, $B^\ell M = z^\ell M \neq 0$ for all ℓ. Nevertheless $\widetilde{M} = 0$. Indeed, M is concentrated in *odd* degrees, so on the charts of Example 4.8 we compute

$$\Gamma\big(D_+(x), \widetilde{M}\big) = \big(M[x^{-1}]\big)_0 = 0, \qquad \Gamma\big(D_+(y), \widetilde{M}\big) = \big(M[y^{-1}]\big)_0 = 0,$$

since x and y annihilate M, and

$$\Gamma\big(D_+(z), \widetilde{M}\big) = \big(M[z^{-1}]\big)_0 \cong \big(\mathbf{k}[z^{\pm 1}](1)\big)_0 = \mathbf{k}[z^{\pm 1}]_1 = 0,$$

because $\deg(z) = 2$, so the ring $\mathbf{k}[z^{\pm 1}]$ has no odd-degree elements. Thus the sheaf $\widetilde{M}$ is locally the zero sheaf. Hence sheafification kills M but $M \notin D_B(S)$. It follows that the induced functor $D(S)/D_B(S) \to D(\mathbb{P}(1,1,2))$ is not an equivalence.

Remark 5.11. Example 5.10 shows that Proposition 5.7(3) fails for simplicial X. The correct simplicial criterion is that $\widetilde{M} = 0$ if and only if $\big(B^\ell M\big)_\alpha = 0$ for $\ell \gg 0$ and all $\alpha \in \mathrm{Pic}(X)$ (e.g. see [CLS11, Exc. 5.3.5]). Only the graded pieces of M in *Picard* degrees are visible to the variety. On $\mathbb{P}(1,1,2)$ we have $\mathrm{Pic}(X) = 2\mathbb{Z}$ (Example 4.8), and the module M above is concentrated in odd degrees, so all of its Picard-degree pieces vanish, thus $\widetilde{M} = 0$.

5.5. **Toric stacks (optional).** Theorem 5.9 fails for $\mathbb{P}(1,1,2)$ because the geometric quotient forgets the finite stabilizers of the G-action, such as the μ_2 at $[0:0:1]$. The failure is repaired not by changing the algebra but by changing the geometry: replace the GIT quotient by the corresponding quotient *stack*. Let X be a simplicial toric variety with Cox ring S and irrelevant ideal B, and let $G = \mathrm{Hom}_{\mathbb{Z}}(\mathrm{Cl}(X), \mathbf{k}^*)$ act on $\operatorname{Spec} S$ as in Theorem 4.6. The toric stack associated to X is the quotient stack

$$\mathcal{X} \;\coloneqq\; \big[\,\mathrm{Spec}(S) \setminus V(B)\,/\,G\,\big].$$

Since X is simplicial, the stabilizers of the G-action are finite, so $\mathcal{X}$ is a smooth toric Deligne–Mumford stack (tame if $\operatorname{char} \mathbf{k}$ does not divide the order of any stabilizer, which is automatic in characteristic zero), and the natural map $\pi\colon \mathcal{X} \to X$ realizes X as its coarse moduli space [BCS05].[2]

A (quasi)coherent sheaf on $[U/G]$ is precisely a G-equivariant (quasi)coherent sheaf on U [Vis89, Ex. 7.21]. Consider $\mathcal{X} = \mathbb{P}_{\mathrm{stack}}(1,1,2) = [\mathbb{A}^3 \setminus \{0\}\,/\,\mathbf{k}^*]$. A quasicoherent sheaf on $\mathbb{A}^3$ is simply an S-module M, and a $\mathbf{k}^*$-equivariant structure on it is a decomposition $M = \bigoplus_{d \in \mathbb{Z}} M_d$ into eigenspaces of the action. That is, exactly a $\mathbb{Z}$-grading compatible with the grading of S. Removing $V(\mathfrak{m})$ kills precisely the modules supported at the origin, i.e. the finite-length modules. Hence

$$\mathrm{coh}\big(\mathbb{P}_{\mathrm{stack}}(1,1,2)\big) \;\simeq\; \mathrm{mod}_{\mathrm{gr}}(S)\,/\,\mathrm{mod}_{\mathfrak{m}}(S),$$

exactly as for $\mathbb{P}^n$. The same reasoning applies in general:

[2] Our viewpoint on toric stacks here is deliberately algebraic, through modules and sheaves. A more geometric treatment via stacky fans and toric morphisms is introduced in [BCS05] and expanded in [GS15a, GS15b].

Theorem 5.12 ([BCS05])**.** *Let X be a simplicial toric variety with $\mathrm{Cl}(X)$-graded Cox ring S and irrelevant ideal B, and let $\mathcal{X} = [\mathrm{Spec}(S) \setminus V(B) \,/\, G]$ be its canonical toric stack. Then*

$$\mathrm{coh}(\mathcal{X}) \;\simeq\; \mathrm{mod}_{\mathrm{gr}}(S) \,/\, \mathrm{mod}_B(S).$$

In short, sheaves on $\mathcal{X}$ are graded S-modules modulo B-torsion, analogous to the smooth case in Proposition 5.7, and this is the whole point of passing from the variety X to the stack $\mathcal{X}$. Passing to derived categories:

Corollary 5.13. *With notation as in Theorem 5.12, there is an equivalence*

$$D(\mathcal{X}) \;\simeq\; D(S) \,/\, D_B(S).$$

When X is smooth, G acts freely on $\mathrm{Spec}(S) \setminus V(B)$, so $\mathcal{X} \cong X$ and Corollary 5.13 recovers Theorem 5.9. When X is simplicial but not smooth, the stack and the variety genuinely differ: for instance $\mathcal{X}$ is smooth with $\mathrm{Pic}(\mathcal{X}) \cong \mathrm{Cl}(X)$, so $\mathcal{O}(1)$ *is* a line bundle on $\mathbb{P}_{\mathsf{stack}}(1,1,2)$, resolving the issues of Remark 5.5.

Remark 5.14 (The variety and the Veronese subring). The singular variety admits a parallel description. Writing $S^{(2)} = \mathbf{k}[x^2, xy, y^2, z]$ for the second Veronese subring of S, generated in degree 2, one has

$$\mathrm{coh}\big(\mathbb{P}(1,1,2)\big) \;\simeq\; \mathrm{mod}_{\mathrm{gr}}\big(S^{(2)}\big) \,/\, \mathrm{mod}_{\mathfrak{m}}\big(S^{(2)}\big),$$

the classical Serre description of $\mathrm{Proj}\, S^{(2)}$. This explains Example 5.10: the module M there is not finite length over S, so it induces a *nonzero* sheaf on the stack, but $M^{(2)} = \bigoplus_{d \in 2\mathbb{Z}} M_d = 0$, so it induces the *zero* sheaf on the variety. The stack quotient is thus tied to S itself, while the GIT quotient is tied to the Veronese subring; for the multigraded module theory of these notes the stack is the more natural object, and we pass to it whenever singular simplicial toric varieties arise.

6. Lecture 6 (Andreas Hochenegger): The Bondal-Thomsen Collection

The Bondal-Thomsen collection is a natural set of line bundles on a toric variety $X = X(\Sigma)$ that can be characterized in several different ways. The following definition is originally due to Bondal [Bon06]. Consider the short exact sequence

$$0 \to M \xrightarrow{\alpha} \mathbb{Z}^n \xrightarrow{\deg} \mathrm{Cl}(X) \to 0, \tag{6.1}$$

where the basis vectors of $\mathbb{Z}^n$ correspond to the rays of Σ. We write α_i for the projection of α onto the i-th basis vector of $\mathbb{Z}^n$, which we identify with the corresponding invariant divisor D_i. Expressed in terms of the pairing between M and N this is $\alpha_i(m) = \langle m, -u_i \rangle$ where u_i is the first integral point on the i-th ray of Σ. By abuse of notation, we again write α_i for the natural extension of α_i to a map $M \otimes \mathbb{Q} \to \mathbb{Q}$.

Further, we define $\lfloor \alpha \rfloor \colon M \otimes \mathbb{Q} \to \mathbb{Z}^n$ by the formula:

$$\lfloor \alpha \rfloor(m) := \sum_i \lfloor \alpha_i(m) \rfloor D_i.$$

Note that though $\alpha(m)$ maps to the trivial element of $\mathrm{Cl}(X) \otimes \mathbb{Q}$, this may not be the case for $\lfloor \alpha \rfloor(m)$.

Definition 6.2. The Bondal-Thomsen collection $\Theta_X \subseteq \mathrm{Cl}(X)$ is the image of $\deg \circ \lfloor \alpha \rfloor$. Elements of Θ_X are called Bondal-Thomsen classes.

It is important to note that Θ_X actually depends only on the Cox ring $k[\mathbb{N}^n]$ with its grading, or equivalently on the rays of the fan Σ, and not on the toric variety X which involves the additional choice of the cones in a fan supported on the rays of Σ. Moreover, as we will see in Theorem 6.6, (2), being Bondal-Thomsen is determined by the image of a class in $\mathrm{Cl}(X) \otimes \mathbb{Q}$ or, equivalently, in $\mathrm{Cl}(X)/\text{torsion}$.

If $m \in M$, then $\lfloor \alpha \rfloor(m) = \alpha(m)$, and this maps to 0 in $\mathrm{Cl}(X)$, so we may consider Θ_X as the image of the induced (nonlinear) map $(M \otimes \mathbb{Q})/M \to \mathrm{Cl}(X)$ which we also call $\lfloor \alpha \rfloor$.

Example 6.3. Consider the case $X = \mathbb{P}^2$. Then $M = \mathbb{Z}^2$ and $\mathbb{Z}^n = \mathbb{Z}^3 = \oplus_{i=0}^2 \mathbb{Z} D_i$, with $\deg(D_i) = 1 \in \mathrm{Cl}(\mathbb{P}^2) = \mathbb{Z}$ for each i.

The map α is represented by the matrix

$$\begin{pmatrix} -1 & 0 \\ 0 & -1 \\ 1 & 1 \end{pmatrix}.$$

Thus

$$\lfloor \alpha \rfloor(a,b) = \lfloor -a \rfloor D_0 + \lfloor -b \rfloor D_1 + \lfloor a+b \rfloor D_2.$$

Identifying $\mathrm{Cl}(\mathbb{P}^2)$ with $\mathbb{Z}$ this becomes

$$\lfloor \alpha \rfloor(a,b) = \lfloor -a \rfloor + \lfloor -b \rfloor + \lfloor a+b \rfloor.$$

Any point in $(M \otimes \mathbb{Q})/M$ can be represented uniquely by an element $(a,b) \in \mathbb{Q}^2$ with $0 \le a, b < 1$. Thus the Bondal-Thomsen collection is the set

$$\lfloor \alpha \rfloor(a,b) = \lfloor -a \rfloor + \lfloor -b \rfloor + \lfloor a+b \rfloor \text{ with } (a,b) \in [0,1)^2$$

This is easy to compute and with $(a,b) \in [0,1)^2$ we have

$$\lfloor \alpha \rfloor(a,b) = \begin{cases} 0 & \text{if } a = b = 0 \\ -2 & \text{if both } a, b \text{ are nonzero but } a+b < 1 \\ -1 & \text{otherwise.} \end{cases}$$

Thus $\Theta_{\mathbb{P}^2} = \{-2, -1, 0\}$.

We can think of the Bondal-Thomsen classes as lattice points in the image under the degree map of the product of $(-1,0]^n \subset \mathbf{R}^n$; such a convex set is called a *zonotope*.

Definition 6.4. The Bondal zonotope is the locus

$$\mathcal{Z} := \left\{ \sum_i a_i \deg(D_i) \;\middle|\; a_i \in (-1,0] \right\} \subseteq \mathrm{Cl}(X) \otimes \mathbf{R}.$$

Example 6.5. The Bondal zonotope of $\mathbb{P}^2$ is the half-open interval $(-3,0] \subset \mathbf{R} = \mathrm{Cl}(\mathbb{P}^2) \otimes \mathbf{R}$.

Theorem 6.6. *The following conditions on the class $d \in \mathrm{Cl}(X)$ of a torus invariant divisor are equivalent:*

(1) d is a Bondal-Thomsen class.
(2) d is equivalent over $\mathbb{Q}$ to a rational divisor of the form $\sum_i a_i D_i$ with $a_i \in (-1,0]$.
(3) $d \otimes 1 \in \mathrm{Cl}(X) \otimes \mathbf{R}$ lies in the Bondal zonotope of X.
(4) In case X is smooth: d represents a line bundle that appears as a direct summand of $F_{\ell,*}\mathcal{O}_X$ for some ℓ, or, equivalently, for every sufficiently divisible ℓ. [3]

Proof. (1) $\Rightarrow$ (2): By definition, a Bondal-Thomsen class has the form $\deg \lfloor \alpha \rfloor(m)$ for some $m \in M \otimes \mathbb{Q}$. Note that $\lfloor \alpha \rfloor(m) - \alpha(m)$ has coefficients in $(-1,0]$. By tensoring (6.1) with $\mathbb{Q}$, we see that $\deg(\alpha(m)) = 0 \in \mathrm{Cl}(X) \otimes \mathbb{Q}$, and thus $\lfloor \alpha \rfloor(m) - \alpha(m)$ is rationally equivalent to the divisor $\lfloor \alpha \rfloor(m)$.

(2) $\Rightarrow$ (1): Conversely, let $D \in \mathbb{Z}^n$ be an integral divisor that is rationally equivalent to a divisor E with coefficients in $(-1,0]$. It follows that $\lfloor D - E \rfloor = D$. But also $D - E \in \ker(\deg \otimes \mathbb{Q})$, so we may write $D - E = \alpha(m)$ for $m \in M \otimes \mathbb{Q}$. Thus $\deg \lfloor \alpha \rfloor(m) = \deg D$ is a Bondal-Thomsen class.

(2) $\iff$ (3): trivial.

(2) $\iff$ (4): We follow the proof given in [Ach15, Theorem 1.1]. First, assume that X is complete. We first show that $F_{\ell,*}\mathcal{O}_X$ is a direct sum of line bundles. Since X has an open affine covering by affine spaces it suffices to treat the case $X = \mathbb{A}^n$, and in this case $F_{\ell,*}\mathcal{O}_X$ is the polynomial ring $k[x_1, ..., x_n]$ regarded as a module over $S := k[x_1^\ell, \ldots, x_n^\ell]$, which is obviously a direct sum of the rank 1 free modules $x^a S$ for all multi-indices a with entries $0 \le a_i < \ell$.

To see that these patch together, note that for any sheaf $\mathcal{L}$ on X the sheaf $F_{\ell,*}\mathcal{L}$ carries a natural action by the group scheme $G := \mathrm{Spec}(k[\mathbb{Z}^n/\ell\mathbb{Z}^n])$, acting by a character. In the case of $\mathbb{A}^n$ the summand $x^a S$ is the eigensheaf for eigenvalue a and it follows that there is no problem patching the summands across different open affine sets.

More generally, $F_{\ell,*}\mathcal{O}_X(b)$ corresponds, when restricted to an open set $\mathbb{A}^n$, to $k[x_1, \ldots, x_n](b)$ regarded as an S-module, and this decomposes in the same way, but with a shift by b.

To prove that the Bondal-Thomsen sheaves appear in the decomposition of $F_{\ell,*}\mathcal{O}_X$ we define exponents $m(e,d)$, for $d, e \in \mathrm{Cl}(X)$, by the formula

$$F_{\ell,*}\mathcal{O}_X(d) = \oplus_e \mathcal{O}_X(e)^{\oplus m(e,d)}.$$

We must show that $m(e,0)$ is nonzero only when e is a Bondal-Thomsen class; and that every Bondal-Thomsen class appears when ℓ is sufficiently divisible.

From the push-pull formula we get

$$F_{\ell,*}\mathcal{O}_X(d) \otimes \mathcal{O}_X(-e) = F_{\ell,*}\mathcal{O}_X(d - \ell e) \tag{1}$$

from which we see that $m(e,d) = m(0, d - \ell e)$ depends only on the difference $d - \ell e$. Thus we define $m(d) := m(0,d)$. The desired conclusion will follow from the case $d = 0$ of the following more general result:

[3] According to [HHL24, Remark 5.4] this can be done in the simplicial case by using the toric stack in place of the variety.

Lemma 6.7. *For $d, e \in \mathrm{Cl}(X)$, the number $m(e,d) = m(d - \ell e)$ is the number of divisors $D = \sum a_i D_i \in \mathbb{Z}^n$ with coefficients a_i in the interval $0 \le a_i < \ell$ such that $\deg D = d - \ell e$.*

Here the D_i are, as always, the natural basis of $\mathrm{Div}(X) = \mathbb{Z}^n$.

Proof. We first assume that X is complete, so that $h^0(\mathcal{O}_X(e))$ is finite-dimensional for all e and the effective divisors on X lie in a pointed cone inside $\mathbb{Z}^n$. We form the generating function

$$h^0(F_{\ell,*}\mathcal{O}_X(d)) := \sum_{e \in \mathrm{Cl}(X)} m(e,d) h^0(\mathcal{O}_X(e)) = \sum_{e \in \mathrm{Cl}(X)} m(d - \ell e) h^0(\mathcal{O}_X(e)).$$

Fixing a basis $\{\mathcal{O}_X(d_i)\}_{i=1,\dots,\rho}$ of $\mathrm{Cl}(X) \cong \mathbb{Z}^\rho$ we define formal power series

$$s(t) = \sum_{a \in \mathbb{Z}^\rho} h^0(\mathcal{O}_X(\sum_i a_i d_i)) t^a, \qquad m(t) = \sum_{a \in \mathbb{Z}^\rho} m(\sum_i a_i d_i) t^a.$$

where, as usual, $a = (a_1, \dots, a_\rho)$ is a multi-index. From equation (6) we get

$$s(t) = m(t) s(t^\ell).$$

Now $h^0(\mathcal{O}_X(d))$ is the number of divisors $D \in \mathrm{Div}(X) = \mathbb{Z}^n$ with $\deg(D) = d \in \mathrm{Cl}(X)$, so

$$s(t) = \sum_{D \in \mathbb{N}^n} t^{\deg D} = \prod_{i=1}^n \frac{1}{1 - t^{\deg D_i}}.$$

Thus

$$m(t) = \frac{s(t)}{s(t^\ell)} = \prod_{i=1}^n \frac{1 - t^{\ell \deg D_i}}{1 - t^{\deg D_i}} = \prod_{i=1}^n (1 + t^{\deg D_i} + \dots + t^{(\ell-1)\deg D_i}).$$

Comparing coefficients, it follows that $m(d)$ is the number of divisors D with $\deg(D) = d \in \mathrm{Cl}(X)$ that have the form $D = \sum a_i D_i$ with $0 \le a_i < \ell$, completing the proof in the case when X is complete.

If X is not complete, we add cones to the associated fan, and then subdivide, to get a birational embedding as an open subvariety $X \to \overline{X}$ where $\overline{X}$ is both complete and smooth. The pushforward functor commutes in an obvious sense with the inclusion. The class group $\mathrm{Cl}(X)$ is the quotient of $\mathrm{Cl}(\overline{X})$ by the classes coming from the new rays, and the restriction of a sum of line bundles on $\overline{X}$ to the open set X and the kernel of the map is generated by the divisors supported on the complement of X; thus the desired formula for X follows from the corresponding formula for $\overline{X}$. $\square$

Conclusion of the Proof of Theorem 6.6
Applying Lemma 6.7 in the case $d = 0$, we see that $m(e, 0)$ is nonzero if and only if there is a divisor D with $\deg D = -\ell e$ such that $D = \sum_i a_i D_i$ with $0 \le a_i < \ell$; that is, $e = -\deg(D/\ell)$, so that e is a Bondal-Thomsen class, and every Bondal-Thomsen class appears for sufficiently divisible ℓ. $\square$

We can use Demazure's Vanishing Theorem [CLS11, Theorem 9.2.3] to derive a useful vanishing theorem related to Bondal-Thomsen classes.

Theorem 6.8 (Demazure Vanishing)**.** *If X is a simplicial, projective toric variety and D is a nef $\mathbb{Q}$-Cartier divisor on X, then $H^p(X, \mathcal{O}_X(\lfloor D \rfloor)) = 0$ for all $p > 0$.*

A similar statement holds for reasonable toric stacks; see [BBB+24, Lemma 2.20]. Note that the round-down depends on expressing D in terms of a basis consisting of torus-invariant Q-divisors; it is not an intrinsic operation on divisors.

Proposition 6.9. *Let $a \in \mathrm{Cl}(X)$ be a nef divisor class. If $-d \in \Theta_X$ is a Bondal-Thomsen class, then $H^p(X, \mathcal{O}_X(a - d)) = 0$ for all $p > 0$.*

Proof. Choose an integral toric divisor $Y = \sum_i \alpha_i D_i$, with $\alpha_i \in \mathbb{Z}$ whose class is $a - d$. Since $-d$ is a Bondal-Thomsen class, we can choose a toric divisor $D = \sum_i \varepsilon_i D_i$, where $\varepsilon_i \in [0, 1)$, whose class is d.

The $\mathbb{Q}$-divisor $Y + D$ is nef because its class is $(a - d) + d = a$. And $\lfloor Y + D \rfloor = Y$ has class $a - d$. By Theorem 6.8,

$$0 = H^p(X, \mathcal{O}_X(\lfloor Y + D \rfloor)) = H^p(X, \mathcal{O}_X(Y)) = H^p(X, \mathcal{O}_X(a - d)).$$

□

Corollary 6.10. *If $-d \in \Theta_X$ is a Bondal-Thomsen class, then $H^i(X, \mathcal{O}_X(-d)) = 0$ for all $i > 0$. If $d \neq 0$, then $H^0(X, \mathcal{O}_X(-d))$ also vanishes.*

Proof. For $p > 0$, we can apply Proposition 6.9 with $a = 0$. When $d \neq 0$, the class $-d$ is the negative of the class of an effective divisor. Since X is complete, the effective divisors lie in a pointed cone, so $\mathcal{O}_X(-d)$ has no global sections. □

Example 6.11 (Frobenius pushforwards)**.** Corollary 6.10 also follows from the Frobenius pushforward definition. For an integer $m > 0$, the toric Frobenius map on X is the finite toric morphism $F_m \colon X \to X$ induced by multiplication by m on N. Thomsen [Tho00] showed that $F_{m*}\mathcal{O}_X$ splits as a direct sum of line bundles. Since F_m is finite,

$$\bigoplus_{\mathcal{O}(E) \subseteq F_{m*}\mathcal{O}_X} H^p\big(X, \mathcal{O}(E)\big) \;=\; H^p\big(X, F_{m*}\mathcal{O}_X\big) \;=\; H^p\big(X, \mathcal{O}_X\big) \;=\; 0 \quad \text{for } p > 0,$$

where the last equality is Theorem 6.8 applied to $D = 0$. Hence every line bundle summand of a toric Frobenius pushforward, i.e. every line bundle whose class is in the Bondal–Thomsen collection, has vanishing higher cohomology.

A similar argument works for the second sentence of Corollary 6.10, regarding H^0.

7. Lecture 7 (Jason McCullough): Constructing the resolution and examples

This section is an introduction to the papers [BCHSY25] and [BH25]. As above, $X = X(\Sigma)$ denotes a simplicial toric variety where Σ is a fan with n rays, and we set $M = N^*$.

A prime ideal generated by elements $x^m - x^p \in S := k[x_1, \ldots, x_n]$, where $m, p \in \mathbb{N}^n$ are multi-indices, in the Cox ring of X, defines a toric subvariety. Any toric subvariety $Y \subset X$ may be defined in this way if the inclusion is induced by a homomorphism from the dense torus of Y to that of X whose image is a subtorus. By [ES96], primeness implies both that the set $L = \{m - p \mid x^m - x^p \in I \cdot k[\mathbb{Z}^n]\}$ is a saturated lattice and that there is a one-to-one correspondence between pure binomial ideals that are prime and saturated lattices meeting $\mathbb{N}^n$ only in 0. Such binomial ideals correspond to affine, not necessarily normal, toric varieties in $\mathbb{A}^n$. The inclusion of dense tori corresponds to the quotient map $M \to M/L$ of character latices.

The ring $R := S/I$ is isomorphic to the semigroup ring $k[\mathbb{N}^n/L]$, where $\mathbb{N}^n/L$ is the image of $\mathbb{N}^n$ under the quotient map $\mathbb{Z}^n \to \mathbb{Z}^n/L$. If $L = M$ then I is the ideal corresponding to the subvariety consisting of the identity element of the dense torus. This example will be treated in the next section.

Theorem 3.4 of [BCHSY25] provides a canonical resolution $\mathbf{F}$ of the normalization $\overline{R}$ of the ring R. It is obtained from a $\mathbb{Z}^n$-graded, L-equivariant S-free resolution $\widetilde{\mathbf{F}}$ of an S-module $K_L \subset k(x_1, \ldots, x_n)$ (Definition 7.2) by factoring out a regular sequence. The relation of $\mathbf{F}$ to $\widetilde{\mathbf{F}}$ is analogous to the relation of a space to its universal cover.

Although the free modules in $\widetilde{\mathbf{F}}$ are not finitely generated, the quotient $\mathbf{F}$ is a finite, S-free, $\mathbb{Z}^n/L$-graded resolution of length equal to the codimension of I, reproving the theorem of Hochster that $\overline{R}$ is Cohen-Macaulay. Moreover, the free modules $\mathbf{F}_i$ have degrees corresponding to the Bondal-Thomsen classes defined in the previous section. It may be easiest to understand the following general definitions while reviewing the special case treated in Example 7.4.

The resolution $\widetilde{\mathbf{F}}$ is built from an infinite L-invariant cellular subdivision of $L_{\mathbf{R}} := \mathbf{R} \otimes L$ and a stratification of $L_{\mathbf{R}}$. Although [BCHSY25] treats a more general stratification, we will treat only the case of the "ceiling stratification" and the L-invariant subdivision of $L_{\mathbf{R}}$ defined by the coordinate hyperplanes $H_{i,j} := \{x \in L_{\mathbf{R}} \mid x_i = j\} \quad j \in \mathbb{Z}$. This divides $L_{\mathbf{R}}$ into cells A whose closures are convex polyhedra, meeting along their faces; see Figure 7.1.

We regard the subdivision above as making $L_{\mathbf{R}}$ into an infinite cell complex. If A is a cell, and $p = (p_1, \ldots, p_n) \in \mathbf{R}^n$ is a point in the interior of A, then we give A the weight

$$\psi(A) := \lceil p \rceil \in \mathbb{Z}^n.$$

This is independent of the choice of p because the subdivision of $L_{\mathbf{R}}$ is induced by the coordinate hyperplanes.

If B is a cell in the boundary of the closure $\overline{A}$, then B is the intersection of $\overline{A}$ with some of the hyperplanes H_j. Thus any interior point of B will be a limit of points in the interior of A at which some coordinates approach an integer. It follows that $\psi(B)$ will be termwise less than or equal to $\psi(A)$; in fact $\psi(A) - \psi(B)$ is a vector of 0s and 1s.

Definition 7.1 (The HHL Complex)**.** The term of $\widetilde{\mathbf{F}}$ corresponding to a cell A is defined to be $S(-\psi(A))$, with

$$\widetilde{\mathbf{F}}_i = \bigoplus_{\dim A = i} S(-\psi(A)).$$

Thus if p is an interior point of A we have

$$-\deg(\psi(A)) = -\deg(\lceil p \rceil) = -\sum_{i=1}^{n} \lceil p_i \rceil D_i = \sum_{i=1}^{n} \lfloor -p_i \rfloor D_i,$$

which is a Bondal-Thomsen class.

We define the differential of the resolution $\widetilde{\mathbf{F}}$ by choosing an orientation on $L_{\mathbf{R}}$ and homogenizing, according to the weights assigned to the chambers, the usual differential of a cell complex $\partial_{L_{\mathbf{R}}} A = \sum \varepsilon_i B_i$, where each $\varepsilon_i = \pm 1$.

With this homogenization, the component of the differential that maps $S(-\psi(A))$ to $S(-\psi(B_i))$ is multiplication by $\varepsilon_i x^{\psi(A)-\psi(B_i)}$, where $\varepsilon_i = \pm 1$, depending on the chosen orientations. The differential preserves the $\mathbb{Z}^n$-grading, the complex $\widetilde{\mathbf{F}}$ is L-equivariant.

In particular,

$$\widetilde{\mathbf{F}}_0 = \bigoplus_{p \text{ a vertex of the cell complex } L_{\mathbf{R}}} S(-\psi(p)),$$

so there is a natural augmentation map from $\widetilde{\mathbf{F}}_0$ to a module K_L generated by the corresponding monomials:

Definition 7.2. $K_L \subset k[\mathbb{Z}^n] \subset k(x_1, \dots, x_n)$ is the S-submodule generated by the monomials

$$\{x^{\psi(p)} \mid p \text{ is a vertex of the cell complex } L_{\mathbf{R}}\}.$$

In Theorem 9.1 we will prove that the complex $\widetilde{\mathbf{F}}$ is a resolution of K_L.

Remark 7.3. Because $\psi(A)$ is termwise $\geq \psi(B)$ when B is a vertex of A, the module K_L contains (and thus is generated by) the monomials $x^{\psi(\mathbf{p})}$ for all points $\mathbf{p} \in L_{\mathbf{R}}$. Thus, while the resolution $\widetilde{\mathbf{F}}$ depends on the cell structure, the module K_L only depends on the lattice L.

Because the differentials of $\widetilde{\mathbf{F}}$ are L-invariant, we can form a quotient by the action of L, obtaining the $\mathbb{Z}^n/L$-graded complex $\mathbf{F}$. Proposition 3.3 of [BCHSY25] shows that the quotient of K_L by this L-action is $\overline{R}$, regarded as a fractional ideal of R. Putting these results together shows that $\mathbf{F}$ is a resolution of $\overline{R}$.

We now describe these resolutions in the two simplest cases: when $I = (x_1 - x_2)$, so that $\overline{R} = R$, and when $I = (x_1^3 - x_2^2)$, so that $\overline{R} = k[x_1, x_2/x_1]$. To simplify the notation, we write $S = k[x, y]$ in place of $k[x_1, x_2]$.

Example 7.4. $R \coloneqq k[x, y]/(x - y)$: In this case the lattice L is generated by the single vector $(1, -1) \in \mathbb{Z}^2$, which corresponds to the monomial x/y if we regard $k[\mathbb{Z}^2]$ as a subset of $k(x, y)$. Since the cellular subdivision of $L_{\mathbf{R}}$ is L-invariant, it suffices to consider its restriction to a fundamental domain for the action of L, which is contained in a single square of the lattice $\mathbb{Z}^2$, as shown in Figure 7.1. In this special case, the construction is the same as that of the "hull resolution" of [BS98, Section 3] (see also [MS04, Sections 9.2–9.3]) of R.

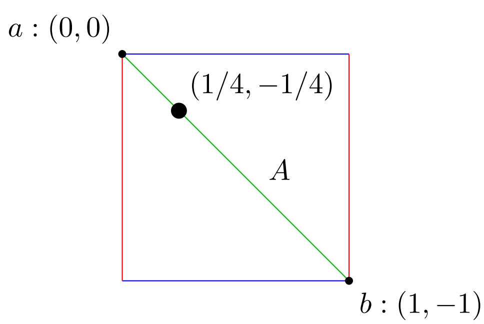


FIGURE 7.1. Diagram for the resolution of $k[x, y]/(x - y)$ on a fundamental domain

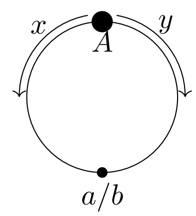


FIGURE 7.2. Diagram for the resolution of $k[x,y]/(x-y)$ on the 1-torus

The set $L_{\mathbf{R}} \subset \mathbf{R}^2$ meets this fundamental domain in the diagonal line A (the dot represents a general point of A, which gives the label of A). There are infinitely many 0-cells, which are L-translates of a, and infinitely many 1-cells, which are the L-translates of A. We choose an L-invariant orientation, and take the differential to be the usual one for a cell complex, homogenized by taking the upper left corner a to have weight $x^0y^0 = 1$, the lower right corner b to have weight x^1y^{-1}, and the cell A to have weight given by the interior point $p = (1/4, -1/4)$, so that A has weight $\psi(p) = (\lceil 1/4 \rceil, \lceil -1/4 \rceil) = (1, 0)$. With these choices $\widetilde{\mathbf{F}}$ and its augmentation to K_L have the form

$$\widetilde{\mathbf{F}} := 0 \to \widetilde{\mathbf{F}}_1 \xrightarrow{\tilde{d}(A)} \widetilde{\mathbf{F}}_0 \to K_L = \sum_{m \in \mathbb{Z}} S \cdot (x^m/y^m) \to 0,$$

where

$$\widetilde{\mathbf{F}}_i = \begin{cases} \oplus_{\ell \in L} S(-Psi(\ell(A))) & \text{if } i = 1 \\ \oplus_{\ell \in L} S(-Psi(\ell(a))) & \text{if } i = 0 \end{cases}$$

If we also write $\ell(A)$ for the generator of $S(\ell(A))$, then the differential $\tilde{d}$ may be written as $\tilde{d}(\ell(A)) = x\,\ell(a) - y\,\ell(b)$, which makes sense because b is the translate of a by a generator of L. Taking the quotient by L we obtain the usual two-term resolution $\mathbf{F} : S(-1) \xrightarrow{d} S$, with differential $d = (x - y)$.

Alternatively, we could form the resolution $\mathbf{F}$ by first taking the quotient of $L_{\mathbf{R}}$ topologically, obtaining a circle, and remembering the monomials associated with each map. This gives a cell structure on a circle with a single 0-cell and a single 1-cell, as in Figure 7.2.

Here is a first example in which $R \neq \overline{R}$.

Example 7.5 (The cuspidal cubic). $R := k[x,y]/(x^3 - y^2) = k[t^2, t^3]$: If $I = (x^3 - y^2)$ then L is generated by the single vector $(3, -2) \in \mathbb{Z}^2$ corresponding to the monomial x^3/y^2. Thus $L_{\mathbf{R}}$ is the diagonal line in Figure 7.3, which shows the closure of a fundamental domain in $L_{\mathbf{R}}$ for the action of L on $L_{\mathbf{R}}$.

The ring R is not integrally closed: the integral closure is $\overline{R} = k[x^2/y] = k[t]$ [4]. Up to translation by L, which identifies a and e, the coordinate hyperplanes cut $L_{\mathbf{R}}$ into four 0-cells a, b, c, d and four 1-cells A, B, C, D shown in Figure 7.3.

[4]The S-free "hull resolution" of R [BS98, Section 3] (see also [MS04, Sections 9.2–9.3]) would be obtained, in this case, from the cell complex with one 1-cell and one 0-cell by considering the whole of the diagonal line in Figure 7.2 as a single 1-cell with weight x^3, and with vertices $(0,0)$ (of weight 1) and $(3,-2)$ (of weight x^3/y^2), yielding the 2-term complex with differential $d = (x^3 - y^2)$. The exposition in the references above differs from that of [BCHSY25] mostly in the Laurent module K_L it describes. The module that [BS98] describes is the S-submodule of $k[\mathbb{Z}^n]$ generated by the lattice L, whereas the module that these notes call K_L has additional generators. To obtain the resolution of $\overline{R}$, [BCHSY25] added the vertices given by the other intersections of the line with the "standard" lattice generated by the vectors corresponding to x and y. Since these intersections do not have integral coordinates, [BCHSY25] introduced the idea of taking the ceilings of their coordinates. If this were a chess game, this move could be marked with "!!".

Following the general scheme described above, we assign weights as follows:

$\psi(A)$: $x^1y^0 = x$
$\psi(B)$: $x^2y^0 = x^2$
$\psi(C)$: $x^2y^{-1} = x^2/y$
$\psi(D)$: $x^3y^{-1} = x^3/y$

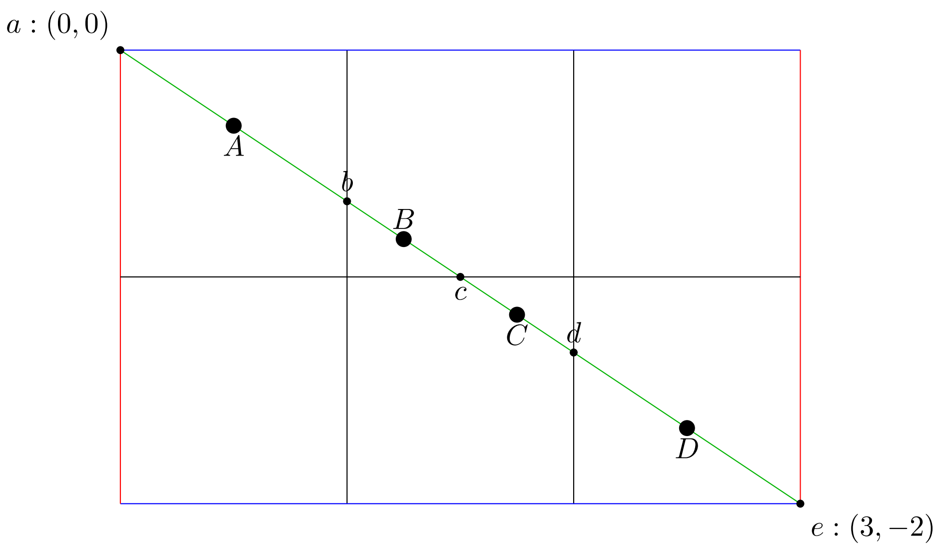
$\psi(a)$: $x^0y^0 = 1$
$\psi(b)$: $x^1y^0 = x$
$\psi(c)$: $x^2y^{-1} = x^2/y$
$\psi(d)$: $x^2y^{-1} = x^2/y$
$\psi(e)$: $x^3y^{-2} = x^3/y^2$

As in the previous example, the complex $\widetilde{\mathbf{F}}$ has just two terms, each of which is an infinite direct sum over the translates by $\ell \in L$; its differential is the sum of the translates of

$$\tilde{d} = \begin{array}{c} \\ a \\ b \\ c \\ d \\ e \end{array}\!\!\begin{array}{c} \begin{array}{cccc} A & B & C & D \end{array} \\ \begin{pmatrix} x & 0 & 0 & 0 \\ -1 & x & 0 & 0 \\ 0 & -y & 1 & 0 \\ 0 & 0 & -1 & x \\ 0 & 0 & 0 & -y \end{pmatrix} \end{array};$$

Again, this makes sense because e is a translate of a.

FIGURE 7.3. Diagram for the resolution of the normalization of $k[x,y]/(x^3-y^2)$



The action of L identifies the vertices a and e. Thus, modulo the action of L the differential of $\mathbf{F}$ is

$$d: \begin{array}{c} \\ a \\ b \\ c \\ d \end{array}\!\!\begin{array}{c} \begin{array}{cccc} A & B & C & D \end{array} \\ \begin{pmatrix} x & 0 & 0 & -y \\ -1 & x & 0 & 0 \\ 0 & -y & 1 & 0 \\ 0 & 0 & -1 & x \end{pmatrix} \end{array}.$$

As before, this quotient can be achieved topologically. Again $L_{\mathbf{R}}/L$ is a circle, with differential given by the red arrows drawn inside the circle in Figure 7.4.

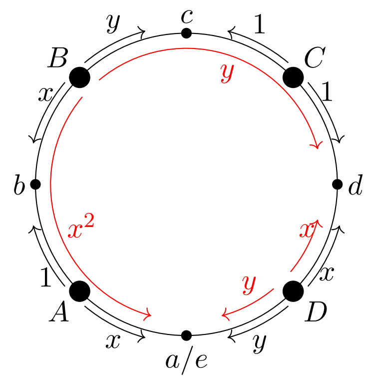


Figure 7.4. Diagram for the resolution of the normalization of $k[x,y]/(x^3-y^2)$ as realized on the torus with both non-minimal and minimal differentials indicated.

This non-minimal matrix is equivalent to the direct sum of the 2×2 identity matrix and the matrix

$$d' := \begin{pmatrix} x^2 & -y \\ -y & x \end{pmatrix}$$

If we give x and y the natural weights 2 and 3, then d' is $\mathbb{Z}^2/L \cong \mathbb{Z}$-homogeneous as a map $S(-4)\oplus S(-3) \to S \oplus S(-1)$.

Taking the quotient of $\widetilde{\mathbf{F}}$ by L is equivalent to setting the action of a generator x^3/y^2 of L to 1, that is, to forming $\widetilde{\mathbf{F}}/(1-x^3/y^2)$.

Since K_L is $\mathbb{Z}^2$-graded, and the elements 1 and x^3/y^2 have different $\mathbb{Z}^2$-degrees, the element $1-x^3/y^2$ is a nonzerodivisor on K_L, and thus the quotient operation preserves exactness.

The determinant of d or d' is (x^3-y^2), so the cokernel is annihilated by this element. Since K_L is generated by the monomials $x^{\psi(p)}$, as p runs over the vertices of $L_{\mathbf{R}}$, $K_L/(1-(x^3/y^2))K_L$ is generated by the monomials $x^{\psi(p)}$ for p in the fundamental domain. These are the fractions $1, x, x^2/y$, which generate the fractional ideal $(1, x^2/y) = (1,t) = \overline{R}$.

Moreover, since $x^3/y = y^2/y = y$ in R, we have

$$(1, x^2/y) * d' = 0$$

and we see that the cokernel of d' is also isomorphic to $R\cdot(x^2/y, y) = R\cdot(x^2/y)$, as required.

8. Lecture 8 (Ryan Watson): More examples and features

The previous lecture focused on the construction of the resolution, and through examples illustrated that for a toric subvariety $X' \subseteq X$ with corresponding lattice $L \subseteq M$, the resolution constructed is a $\mathbb{Z}^n/L$-graded resolution of the normalization $\overline{R}$ of the ring S/I_L. The proof of this is deferred to Section 9. In this section, we describe in more detail the structure of the resolution as it relates to the Bondal-Thomsen classes. We start with the following summary of the resolution and a general statement about its relationship with the Bondal-Thomsen collection.

Corollary 8.1. *Let $X' \subseteq X$ be a toric subvariety, let $L \subseteq M$ be the corresponding sublattice, and let I_L be the corresponding binomial ideal. There is an S-free $\mathbb{Z}^n/L$-graded resolution $\mathbf{F}$ of the normalization $\overline{R}$ of the ring S/I_L whose free modules are given by $\mathbf{F}_i = \oplus S(-\deg\psi(A))$, where the sum is over all cells A of $(L\otimes\mathbf{R})/L$ having dimension i. Moreover, each $S(-\deg\psi(A))$ is a*

A single fundamental domain is as follows:

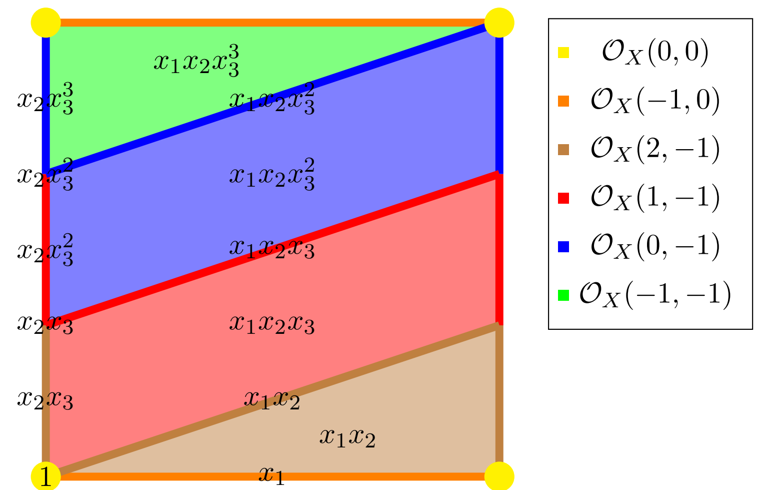


FIGURE 8.1. A diagram on the closure of a single fundamental domain for the stratification of $L_{\mathbf{R}}$ corresponding to $\mathcal{H}_3$, the third Hirzebruch surface. Each cell of the fundamental domain is labeled with a monomial, An actual fundamental domain omits the closed top and right edges.

Bondal-Thomsen class. The construction depends only on the Cox ring with its grading, and not on the additional data of the irrelevant ideal of the toric variety.

Proof. The Bondal-Thomsen assertion holds because $\psi = \lceil \alpha \rceil$, where $\alpha : M \to \mathbb{Z}^n$ is the inclusion given in Lecture 6. The other statements about the construction follow at once from Definition 7.1. The statement that the complex gives a resolution of the normalization is proven in Theorem 9.1. $\square$

Example 8.2. If $L = M$, then, since the cell decomposition of the linear space $M_{\mathbf{R}}$ is defined by the coordinate hyperplanes, every point of $M_{\mathbf{R}} \setminus M$ is the interior point of some cell. Thus all the Bondal-Thomsen classes appear in the resolution $\mathbf{F}$ of Corollary 8.1.

From now on we will omit α, and consider L as a sublattice of M, and M as a sublattice of $\mathbb{Z}^n$.

We can illustrate that the summands are Bondal-Thomsen elements in the case of the Hirzebruch surface by observing that the map $\psi : L_{\mathbf{R}} \to \mathbb{Z}^n$ stratifies $L_{\mathbf{R}}$ in an L-equivariant manner into strata that correspond exactly to the Bondal-Thomsen elements.

Here the Cox ring of the Hirzebruch surface is $S = k[x_1, x_2, x_3, x_4]$ graded by $\mathbb{Z}^2$ with $\deg(x_1) = \deg(x_3) = (1,0)$, $\deg(x_2) = (-3,1)$ and $\deg(x_4) = (0,1)$. Since $L = M$ here, we have $L \cong \mathbb{Z}^2$. Moreover, $L = M$ as a sublattice of $\mathbb{Z}^4$ is spanned by the basis vectors $v = (1,0,-1,0)$ and $w = (0,1,3,-1)$. In Figure 8.1, we depict $L_{\mathbf{R}}$ so that these two basis vectors are aligned with the x- and y-axes, respectively.

The diagram has been labeled with monomials; for example, one point on the interior of the bottom edge is $(1/2,0)$ and thus it corresponds to the free module $S(-\lceil (1/2,0) \rceil) = S(-1,0)$, which we have labeled by the monomial x_1. If we stratify the cells of $L_{\mathbf{R}}$ by degree in $\mathrm{Cl}(X)$, then each of the six Bondal-Thomsen elements gives rise to a stratum. The cells in this fundamental domain have been colored according to the BT collection elements, which in this case are $\mathcal{O}_X(0,0)$, $\mathcal{O}_X(-1,0)$, $\mathcal{O}_X(2,-1)$, $\mathcal{O}_X(1,-1)$, $\mathcal{O}_X(0,-1)$, and $\mathcal{O}_X(-1,-1)$.

The fact that the resolution $\mathbf{F}$ in this construction does not depend on the irrelevant ideal of S suggests a uniformity among the toric varieties having the same graded Cox ring. This is the starting point for the study of the Cox category, beginning in Lecture 12.

Here is an important special case:

Corollary 8.3. *If X and X' are toric varieties with the same graded ring S as Cox ring, then the HHL complex of free $S \otimes_k S$-modules resolving the closed graph of the identity $X \subseteq X \times X$ is equal to the complex associated with the graph $X \subseteq X \times X'$ of the birational map from X to X' that restricts to the identity on the dense torus. Moreover, the summands of the free modules will all be of the form $S(d) \boxtimes S(d')$ for certain pairs of Bondal-Thomsen classes d, d'.*

Proof. Let T be the common dense torus of X and X'. The ideal of the diagonal in $T \times T$ is generated by the binomials

$$\{x^a \otimes 1 - 1 \otimes x^a \mid a \in M\} \subset k[x_1^{\pm 1}, \ldots, x_n^{\pm 1}] \otimes k[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$$

which corresponds to the sublattice

$$L = \{(v, -v) \mid v \in M\} \subset M \times M.$$

where, as usual, we have suppressed the explicit embedding of L as a sublattice of M. This is the same in both cases of the corollary. □

Not every pair of Bondal-Thomsen classes will appear in the resolutions described in Corollary 8.3. Nevertheless, the following statement holds:

Corollary 8.4. *Every Bondal-Thomsen class appears both as the first and as the second member of a pair (d, d') that appears in the HHL resolution of the diagonal of X.*

Proof. The lattice

$$L := \{(\mathbf{v}, -\mathbf{v}) \in \mathbb{Z}^{2n} \,|\, \mathbf{v} \in M_X\}.$$

in the proof above projects surjectively onto each factor, and the cell decomposition of $L_{\mathbf{R}}$ induced by the embedding into $\mathbb{Z}^n \times \mathbb{Z}^n$ is the same as that induced by the embedding into $\mathbb{Z}^n$.

Thus the summands that appear in the resolution of the diagonal are of the form $\mathcal{O}_{X \times X}(-\lceil(\mathbf{p}, -\mathbf{p})\rceil)$ for $\mathbf{p}$ a point in $\mathbf{R} \otimes M_X$ which can then be expanded as follows:

$$\mathcal{O}_{X \times X}(-\lceil(\mathbf{p}, -\mathbf{p})\rceil) = \mathcal{O}_X(-\lceil\mathbf{p}\rceil) \boxtimes \mathcal{O}_X(\lfloor\mathbf{p}\rfloor).$$

The left hand side of this summand is $\mathcal{O}_X(D)$ for some D in the BT collection by the same reason as in Corollary 8.1. And $\mathcal{O}_X(\lfloor\mathbf{p}\rfloor)$ is quite directly the generator of the BT collection associated to $\mathbf{p}$. Finally, as $\mathbf{p}$ varies over all points in $\mathbf{R} \otimes M_X$, every Bondal-Thomsen element appears as the Bondal-Thomsen collection was defined as the image of M_X under the map described in Definition 6.2. □

The Fourier-Mukai transform with kernel equal to a resolution of the diagonal, a key tool in the proof of the main theorem of [BES20], takes a sheaf $\mathcal{F}$ to a monad for $\mathcal{F}$ (that is, a complex whose only homology is $\mathcal{F}$). We will see in Construction 10.5 that the terms of this complex are direct sums of sheaves drawn from the sheafified left-hand factors in the HHL resolution of the diagonal. Consequently, applying the Fourier-Mukai transform to arbitrary coherent sheaves yields monads whose summands are all Bondal-Thomsen elements, as follows:

Corollary 8.5. *If $\mathcal{F}$ is a coherent sheaf on X, then it is the homology of a monad whose summands are all Bondal-Thomsen elements of X.* □

Remark 8.6. The HHL resolution of the diagonal is minimal as a complex of free S-modules. This does not necessarily hold for other L.

We can now describe the resolution of the diagonal in the case when X is the Hirzebruch surface $\mathcal{H}_3$.

Example 8.7. If $X = \mathcal{H}_3$, and S is the Cox ring of X, then each summand of a free module in the HHL complex has the form $S(-\deg(\psi(A))) \boxtimes S(-\deg(\psi(-A)))$. As an example, for the 2-dimensional cell A corresponding to the bottom-right triangle of Figure 8.1, the corresponding Bondal-Thomsen element is $\mathcal{O}_X(2,-1)$. The cell $-A$ is equivalent after a translation by an element of L to the top-left triangle in the diagram, which corresponds to the Bondal-Thomsen element $\mathcal{O}_X(-1,-1)$. This gives the summand $S(2,-1) \boxtimes S(-1,-1)$ appearing in homological degree 2.

Repeating this construction for every cell gives all the summands of the resolution of the diagonal.

$$\mathbf{F}: \begin{matrix} S(0,0)\boxtimes S(0,0) \\ \oplus \\ S(2,-1)\boxtimes S(1,-1) \\ \oplus \\ S(1,-1)\boxtimes S(2,-1) \end{matrix} \longleftarrow \begin{matrix} S(-1,0)\boxtimes S(-1,0) \\ \oplus \\ (S(2,-1)\boxtimes S(0,-1))^2 \\ \oplus \\ (S(1,-1)\boxtimes S(1,-1))^2 \\ \oplus \\ (S(0,-1)\boxtimes S(2,-1))^2 \end{matrix} \longleftarrow \begin{matrix} S(2,-1)\boxtimes S(-1,-1) \\ \oplus \\ S(1,-1)\boxtimes S(0,-1) \\ \oplus \\ S(0,-1)\boxtimes S(1,-1) \\ \oplus \\ S(-1,-1)\boxtimes S(2,-1) \end{matrix} \longleftarrow 0$$

9. Lecture 9 (Greg Smith): The proof

Theorem 9.1. [HHL24] *With notation as in Section 7, the complexes $\widetilde{\mathbf{F}}$ and $\mathbf{F}$ are S-free resolutions of K_L and $\overline{S/I_L}$, respectively.*

The original proof of the Hanlon-Hicks-Lazarev Theorem in [HHL24] is fairly technical. However, two recent preprints have dramatically simplified the proof. The preprint [BCHSY25] (see in particular Proposition 3.2) relates the Hanlon-Hicks-Lazarev construction to a more classical algebraic construction involving binomial ideals, and provides an elementary and purely algebraic proof of the main result, as well as a broad generalization of this result. The preprint [BH25] utilizes a similar basic idea as [BCHSY25], but provides a terser and narrower proof.

Proof. The proof has two parts: the acyclicity of $\mathbf{F}$ and $\widetilde{\mathbf{F}}$ (that is the statement that $H_i = 0 for i >$ 0; and the computation of their homology H_0.

9.1. **Acyclicity.** [5] Because $\widetilde{\mathbf{F}}$ is L-equivariant, it may be regarded as a free graded complex over the ring

$$S[L] = S \cdot \{z^{\mathbf{u}} \mid \mathbf{u} \in L\} \subseteq S[z_1^{\pm 1}, \dots, z_n^{\pm 1}],$$

where $z^{\mathbf{u}}$ acts on a monomial $x^{\mathbf{v}} \in K_L$ by sending it to $x^{\mathbf{v}+\mathbf{u}}$. Similarly, K_L can be regarded as an $S[L]$-module, and in this view it is a finitely generated $S[L]$-module because there are only finitely many vertices in a fundamental domain for the action of L on $L_{\mathbf{R}}$. The degrees of the generators of $\widetilde{\mathbf{F}}_0$ are the same as those of K_L, so there is a natural augmentation $\widetilde{\mathbf{F}}_0 \to K_L$, and it is easy to check that this composes to 0 with the differential of $\widetilde{\mathbf{F}}$.

The ring S is a homomorphic image of $S[L]$; indeed,

$$S = S[L]/(\{z^{\mathbf{u}} - 1 \mid \mathbf{u} \in L\}).$$

If P is any L-graded $S[L]$-module, and $0 \neq \mathbf{u} \in L$, then $1 - z^{\mathbf{u}}$ is a nonzerodivisor on P because p and $z^{\mathbf{u}}p$ have distinct degrees. It follows easily by induction that if $\mathbf{u}_1, \dots, \mathbf{u}_r \in L$ is a basis, then

[5] The following proof is derived from that of Proposition 2.13 and the alternative proof sketch of Corollary 2.15 in [BCHSY25] and is similar to the proof of Proposition 2.6 in [BH25].

the sequence $z^{\mathbf{u}_1}-1,\dots,z^{\mathbf{u}_r}-1$ is a regular sequence on $S[L]$ and also on the module K_L. The complex $\mathbf{F}$, which we formerly described as the quotient of $\widetilde{\mathbf{F}}$ by L, has the form

$$\mathbf{F}=\widetilde{\mathbf{F}}/(\{z^{\mathbf{u}}-1\mid \mathbf{u}\in L\})\widetilde{\mathbf{F}}=\tilde{\mathbf{F}}\otimes_{S[L]}S.$$

This is a $\mathbb{Z}^n/L$-graded complex of finitely generated free S-modules.

As a k-vector space the graded complex $\widetilde{\mathbf{F}}$ is the direct sum of the subcomplexes $\widetilde{\mathbf{F}}_{\mathbf{u}}$ consisting of elements of degree $\mathbf{u}$, called the $\mathbf{u}$-strand of the complex. Similarly $\mathbf{F}$ is the direct sum of its $\overline{\mathbf{u}}$-strands. Thus $\widetilde{\mathbf{F}}$ and $\mathbf{F}$ are acyclic if and only if the same is true of each strand.

Fix a degree $\mathbf{u}\in\mathbb{Z}^n$, and let

$$(L_{\mathbf{R}})_{\le\mathbf{u}}:=\{\mathbf{p}\in L_{\mathbf{R}}\mid\psi(\mathbf{p})\le\mathbf{u}\}=\{\mathbf{p}\in L_{\mathbf{R}}\mid\lceil p_i\rceil\le u_i \text{ for all } i=1,\dots,n\}$$

For each cell $A\subseteq L_{\mathbf{R}}$ the summand corresponding to A in $\widetilde{\mathbf{F}}$ has component in degree $\mathbf{u}$ given by

$$(*)\qquad S(-\psi(A))_{\mathbf{u}}=\begin{cases}k & \text{if } x^{\psi(A)} \text{ divides } x^{\mathbf{u}}\\ 0 & \text{otherwise}\end{cases}.$$

Thus the cells of $L_{\mathbf{R}}$ whose corresponding summands contribute to the $\mathbf{u}$-strand are exactly the cells that are contained in $(L_{\mathbf{R}})_{\le\mathbf{u}}$. Moreover, the maps between these summands are the cellular differentials, so we have

$$\tilde{\mathbf{F}}_{\mathbf{u}}\cong C_\bullet((L_{\mathbf{R}})_{\le\mathbf{u}};k)$$

.

Finally, $(L_{\mathbf{R}})_{\le\mathbf{u}}$ is a polyhedron and thus contractible or empty. As a consequence, the only homology is in H_0, completing the proof of acyclicity.

9.2. $H_0(\widetilde{\mathbf{F}})=K_L$ **and** $H_0(\mathbf{F})\cong\overline{S/I_L}$**.** The monomial module K_L is the S-module generated by the monomials $x^{\mathbf{u}}$ as $\mathbf{u}$ ranges over the 0-dimensional faces of the complex L_R. These are, by definition, the degrees of the free generators of $\widetilde{\mathbf{F}}_0$, so there is a canonical $\mathbb{Z}^n$-graded map $\tilde{\mathbf{F}}_0\to k[\mathbb{Z}^n]$ whose image is K_L. Moreover, the kernel of this map contains the image of $\widetilde{\mathbf{F}}_1$, so the quotient $\widetilde{\mathbf{F}}_0\to H_0(\widetilde{\mathbf{F}})$ lifts to a surjective $\mathbb{Z}^n$-graded map $H_0(\widetilde{\mathbf{F}}_0)\to K_L$. To show that this map is an isomorphism, it suffices to show that $H_0(\tilde{\mathbf{F}})_{\mathbf{u}}\cong(K_L)_{\mathbf{u}}$. For a degree $\mathbf{u}\in\mathbb{Z}^n$, $H_0(\tilde{\mathbf{F}})_{\mathbf{u}}=H_0((L_{\mathbf{R}})_{\le\mathbf{u}})$. Moreover $(L_{\mathbf{R}})_{\le\mathbf{u}}$ is contractible if there exists a vertex $\mathbf{v}\in L_{\mathbf{R}}$ such that $\psi(\mathbf{v})$ divides $x^{\mathbf{u}}$, and empty otherwise. As K_L is generated by the vertices $\mathbf{v}\in L_{\mathbf{R}}$, it follows that $(L_{\mathbf{R}})_{\le\mathbf{u}}$ is contractible if and only if $x^{\mathbf{u}}\in K_L$ and empty otherwise.

It remains to show that $K_L\otimes_{S[L]}S\cong\overline{S/I_L}$. [6] The ring S/I_L is the semigroup ring of the semigroup A that is the image of $\mathbb{N}^n$ under the map $\mathbb{Z}^n\to\mathbb{Z}^n/L$. The integral closure of a semigroup ring $k[A]$ is the semigroup ring of the saturation $\overline{A}$ of A, which consists of the elements in $\{a\in\mathbb{Z}^n/L\}$ such that $ma\in A$ for some $m\in\mathbb{N}$, or equivalently [Eis95, ***]

$$\overline{A}:=(\mathbf{R}_{\ge0}A)\cap(\mathbb{Z}^n/L)$$

Quite generally, if $B\subset A$ is an inclusion of modules over a ring R and $x\in R$ is a nonzerodivisor on $M=A/B$, then $\operatorname{Tor}_1^R(R/x,M)=0$. Thus, from the long exact sequence in Tor we see that the inclusion $B\subset A$ induces an inclusion $B/xB\subset A/xA$. It follows by induction that, as S is a quotient of $S[L]$ the elements z^u-1 which is also a regular sequence on the graded $S[L]$-module $k[\mathbb{Z}^n]/K_L$ the inclusion $K_L\hookrightarrow k[\mathbb{Z}^n]$ induces an inclusion $K_L\otimes_{S[L]}S\hookrightarrow k[\mathbb{Z}^n]\otimes_{S[L]}S\cong k[\mathbb{Z}^n/L]$. Viewed as a submodule of $k[\mathbb{Z}^n/L]$ under this inclusion, the module $K_L\otimes_{S[L]}S$ is generated by monomials of the form $x^{\lceil\mathbf{v}\rceil}$ for $\mathbf{v}$ a vertex of the cellular structure on $L_{\mathbf{R}}$. Since the normalization,

[6]We follow the approach of Proposition 3.3 in [BCHSY25]. The corresponding result in [BH25] is Lemma 2.8.

viewed as the semigroup ring $k[\overline{A}]$, is naturally an S-submodule of $k[\mathbb{Z}^n/L]$, it remains to show that these two submodules are identical.

First we show that the classes of the monomials $x^{\lceil \mathbf{v} \rceil}$ belong to the semigroup $\overline{A}$. Let $\eta : \mathbb{Z}^n \to \mathbb{Z}^n/L$ be the quotient map and $\eta_{\mathbf{R}} : \mathbf{R}^n \to \mathbf{R}^n/L$ the corresponding map obtained by tensoring with $\mathbf{R}$. Because A contains representatives of all monomials in S, we have $\mathbf{R}_{\geq 0}A = \eta_{\mathbf{R}}(\mathbf{R}^n_{\geq 0})$.

For a vertex $\mathbf{p} \in L_{\mathbf{R}}$, we have

$$\eta(\lceil \mathbf{p} \rceil) = \eta(\lceil \mathbf{p} \rceil) - \eta_{\mathbf{R}}(\mathbf{p}) = \eta_{\mathbf{R}}(\lceil \mathbf{p} \rceil - \mathbf{p})$$

However, $\lceil \mathbf{p} \rceil - \mathbf{p}$ has all non-negative coordinates and so the corresponding monomial is in $\mathbf{R}_{\geq 0}A = \eta_{\mathbf{R}}(\mathbf{R}^n_{\geq 0})$.

For the converse, suppose that $\mathbf{a} \in \overline{A}$. We wish to show that the corresponding element of $k[\overline{A}]$ is an element of $K_L \otimes_{S[L]} S$. As $\mathbf{a} \in \overline{A} = \eta(\mathbf{R}^n_{\geq 0}) \cap (\mathbb{Z}^n/L)$, there exist $\mathbf{q} \in \mathbf{R}^n_{\geq 0}$ and $\mathbf{u} \in \mathbb{Z}^n$ such that

$$\mathbf{a} = \eta_{\mathbf{R}}(\mathbf{q}) = \eta(\mathbf{u})$$

It suffices to prove that there exists a vertex $\mathbf{p}' \in L_{\mathbf{R}}$ such that $x^{\psi(\mathbf{p}')}$ divides $x^{\mathbf{u}}$. Let $\mathbf{p} = \mathbf{u} - \mathbf{q}$. Since $\lceil \mathbf{p} \rceil = \mathbf{u} - \lfloor \mathbf{q} \rfloor$ and $\mathbf{q} \in \mathbf{R}^n_{\geq 0}$, it follows that $x^{\psi(\mathbf{p})}$ divides $x^{\mathbf{u}}$. To get a vertex, observe that if $\mathbf{p}'$ is any vertex of the unique cell of $L_{\mathbf{R}}$ that contains $\mathbf{p}$ in its relative interior, then $x^{\psi(\mathbf{p}')}$ divides $x^{\psi(\mathbf{p})}$ and thus divides $x^{\mathbf{u}}$ as desired.

This completes the proof of Theorem 9.1. $\square$

Example 9.2. Building on Example 7.5, where $n = 2$ and $L = \mathbb{Z} \cdot (3, -2)$, we illustrate the proof that $K_L \otimes_{S[L]} S \cong \overline{S/I}$.

We can indentify $\mathbb{Z}^2/L$ with $\mathbb{Z}$ by the isomorphism $\gamma : (a, b) \mapsto 2a + 3b$. With this identification, the semigroup A is the generated by 2 and 3 in $\mathbb{Z}$ and it's saturation $\overline{A}$ is $\mathbb{N}$. Moreover, $\mathbb{N}$ is generated as a module over A by the element 1.

To prove that 1 is in the image of K_L, consider the vertex $d = (2, -4/3)$ of the induced cell structure in Figure 7.3. The Laurent monomial associated to this point by ψ is x^2y^{-1}. This corresponds to the vector $(2, -1)$, and $\gamma(2, -1) = 1$, and so we get the generator of the semigroup in this way.

To follow the steps of the proof given above observe that $(2, -1) - (2, \frac{-4}{3}) = (0, \frac{1}{3})$. This tells us that

$$\gamma(2, -1) = 0 \cdot \gamma(1, 0) + \frac{1}{3} \cdot \gamma(0, 1) = \frac{1}{3} \cdot 3 = 1.$$

The reverse direction is more subtle. Starting with the same generator 1 of $\overline{A}$, we work backwards. As there is not in general a unique vertex of $L_{\mathbf{R}}$ which realizes this generator, there will be choices involved. First we choose some way of presenting 1 in terms of 2 and 3 with $\mathbf{R}_{\geq 0}$ coefficients, for example $1 = \frac{1}{2} \cdot 2$. Following the notation from the proof, $\mathbf{q} = (\frac{1}{2}, 0)$. Let us still choose $\boldsymbol{u} = (2, -1)$. The goal now is to find a point $\boldsymbol{p}$ such that $\lceil \boldsymbol{p} \rceil$ is componentwise $\leq (2, -1)$. The proof suggests the following difference:

$$\boldsymbol{p} = \boldsymbol{u} - \boldsymbol{q} = (\frac{3}{2}, -1).$$

As $\lceil \boldsymbol{p} \rceil = (2, -1)$, we exactly recover $(2, -1)$, but the more interesting part of the proof is to realize that $\boldsymbol{p} \in L_{\mathbf{R}}$. Here this follows as $\boldsymbol{p} = \frac{1}{2}(3, -2)$.

In this case, the point $\boldsymbol{p}$ is exactly the vertex c from Figure 7.3. Moreover, as we vary the choice of $\boldsymbol{q}$, we get all points $\boldsymbol{p}$ on the line segment from c to d in Figure 7.3.

10. Lecture 10 (Jay Yang): Applications of the Hanlon-Hicks-Lazarev Theorem

Let X be a smooth projective toric variety, and let $\mathcal{O}_\Delta$ denote the structure sheaf of the diagonal subvariety $\Delta \hookrightarrow X \times X$. The following is a consequence of the Hanlon-Hicks-Lazarev Theorem:

Theorem 10.1. *There exists a locally free resolution $\mathcal{R}$ of $\mathcal{O}_\Delta$ of the diagonal $\Delta_X \subseteq X \times X$ of length $\dim X$ whose terms are direct sums of line bundles in the Thomsen collection of $X \times X$.*

Remark 10.2. The elements of the Thomsen collection of $X \times X$ are of the form $\mathcal{L} \boxtimes \mathcal{L}'$, where $\mathcal{L}$ and $\mathcal{L}'$ are in the Thomsen collection of X.

Theorem 10.1 led to proofs of three conjectures in commutative algebra and toric geometry.

10.1. The Berkesch-Erman-Smith Conjecture.

Let S be the Cox ring of X, and let M be an S-module. In [BES20], Berkesch-Erman-Smith define a virtual resolution of M to be a complex F of free S-modules such that $\widetilde{F}$ is a locally free resolution of $\widetilde{M}$. They posed the following conjecture:

Conjecture 10.3 (Berkesch-Erman-Smith's Conjecture)**.** *Every finitely generated $\operatorname{Pic} X$-graded S-module M admits a virtual resolution of length at most $\dim X$.*

In fact, this was originally posed as a question [BES20, Question 6.5], but it has been stated as a conjecture in subsequent works, e.g. [Yan21, BS24]. Before Hanlon-Hicks-Lazarev proved Conjecture 10.3, it was known when X is a product of projective spaces [BES20], when $\operatorname{Pic}(X)$ has rank 2 [BS24], and when M is a monomial ideal [Yan21].

Let us explain how Theorem 10.1 implies Conjecture 10.3. We will need the following:

Theorem 10.4 (Fujita Vanishing Theorem [Laz04, Theorem 1.4.35])**.** *Let $\mathcal{F}$ be a coherent sheaf on X, and let $\mathcal{A}$ be an ample line bundle on X. There exists $m \gg 0$, depending on $\mathcal{F}$ and $\mathcal{A}$, such that*

$$H^j(X, \mathcal{F} \otimes \mathcal{A}^m \otimes \mathcal{L}') = 0$$

for all $j > 0$ and all nef line bundles $\mathcal{L}'$ on X.

The following construction gives a particularly useful expression of Fourier-Mukai transforms with respect to the kernel $\mathcal{R}$, where $\mathcal{R}$ is a resolution of the diagonal as in Theorem 10.1:

Construction 10.5. Let $\mathcal{F}_1$ and $\mathcal{F}_2$ be coherent sheaves on X, where $\mathcal{F}_2$ is locally free. Let $\pi_1, \pi_2 \colon X \times X \to X$ be the projection maps. By the projection formula and base change, we have canonical isomorphisms

$$\mathbf{R}\pi_{2*}(\mathcal{F}_1 \boxtimes \mathcal{F}_2) \cong \mathbf{R}\pi_{2*}\pi_1^*(\mathcal{F}_1) \otimes_{\mathcal{O}_X} \mathcal{F}_2 \cong \mathbf{R}\Gamma(X, \mathcal{F}_1) \otimes_k \mathcal{F}_2$$

in $D(X)$. We now use this to compute the Fourier-Mukai transform of a coherent sheaf $\mathcal{F}$ on X with respect to the kernel $\mathcal{R}$. Given $\mathcal{G} \in \operatorname{coh}(X)$, let $\mathcal{C}_\mathcal{G}$ denote the Čech complex of $\mathcal{G}$ associated to the affine open cover of X arising from the maximal cones in its fan. Our Fourier-Mukai transform of $\mathcal{F}$ is isomorphic, in $D^b(X)$, to the totalization of the following bicomplex, where the horizontal maps are induced by the differentials in $\mathcal{R}$, the vertical maps are induced by the Čech differentials, and "$\mathcal{L}_1 \boxtimes \mathcal{L}_2 \in \mathcal{R}_i$" is shorthand for "$\mathcal{L}_1 \boxtimes \mathcal{L}_2$ is a summand of $\mathcal{R}_i$":

$$0 \longleftarrow \bigoplus_{\mathcal{L}_1 \boxtimes \mathcal{L}_2 \in \mathcal{R}_0} \mathcal{C}_{\mathcal{F} \otimes \mathcal{L}_1} \otimes \mathcal{L}_2 \longleftarrow \cdots \longleftarrow \bigoplus_{\mathcal{L}_1 \boxtimes \mathcal{L}_2 \in \mathcal{R}_{\dim X}} \mathcal{C}_{\mathcal{F} \otimes \mathcal{L}_1} \otimes \mathcal{L}_2 \longleftarrow 0\,. \tag{10.6}$$

Since the differentials of $\mathcal{C}_\mathcal{G}$ have entries in $\mathbf{k}$, the columns of (10.6) split. Thus, we may apply [EFS03, Lem. 3.5] to conclude that the totalization of (10.6) is homotopy equivalent to a complex $\mathcal{B}(\mathcal{F})$ concentrated in degrees $k = -\dim X, \ldots, \dim X$ with terms

$$\mathcal{B}(\mathcal{F})_k = \bigoplus_{i-j=k} \bigoplus_{\mathcal{L}_1 \boxtimes \mathcal{L}_2 \in \mathcal{R}_i} H^j(X, \mathcal{F} \otimes \mathcal{L}_1) \otimes \mathcal{L}_2. \tag{10.7}$$

Of course, the homology of $\mathcal{B}(\mathcal{E})$ is concentrated in degree 0 and isomorphic to $\mathcal{F}$.

Proof of Conjecture 10.3. Let $\mathcal{F}$ be the sheaf on X associated to our S-module M, and let $\mathcal{A}$ be an ample line bundle on X. Applying Theorem 10.4 to the pair $\mathcal{F} \otimes \mathcal{L}$ and $\mathcal{A}$ for all the (finitely many) line bundles $\mathcal{L}$ in the Thomsen collection of X (and taking $\mathcal{L}' = \mathcal{O}$), we may choose $m \gg 0$ such that $H^j(X, \mathcal{F} \otimes \mathcal{L} \otimes \mathcal{A}^m) = 0$ for all $j > 0$. Let $\mathcal{B}(\mathcal{F})$ be as in (10.7). We have $\mathcal{B}(\mathcal{F} \otimes \mathcal{A}^m)_\ell = 0$ unless $0 \leq \ell \leq \dim X$, and so $\mathcal{B}(\mathcal{F} \otimes \mathcal{A}^m) \otimes \mathcal{A}^{-m}$ is a locally free resolution of $\mathcal{F}$ by sums of line bundles of length at most $\dim X$. Taking global sections of (all twists of) $\mathcal{B}(\mathcal{F} \otimes \mathcal{A}^m) \otimes \mathcal{A}^{-m}$ gives the desired virtual resolution. □

10.2. **Bondal's Conjecture.** Given a triangulated category $\mathcal{T}$, we say objects $T_1, \dots, T_m \in \mathcal{T}$ classically generate $\mathcal{T}$ if the smallest triangulated subcategory of $\mathcal{T}$ that is closed under taking summands and that contains $T_1, \dots T_m$ is $\mathcal{T}$ itself.

Conjecture 10.8 (Bondal's Conjecture)**.** *The line bundles in the Thomsen collection of X classically generate $D^b(X)$.*

In fact, Bondal stated without proof in [Bon06] that Conjecture 10.8 holds. But this statement was later referred to as Bondal's Conjecture in e.g. [Ueh14, HHL24]. See [HHL24, Remark 1.5] for a list of known cases of Bondal's Conjecture before it was proven by Hanlon-Hicks-Lazarev.

Proof of Conjecture 10.8. Given a coherent sheaf $\mathcal{E}$ on X, the complex $\mathcal{B}(\mathcal{E})$ from the proof of Conjecture 10.3 exhibits $\mathcal{E}$ as an iterated extension in $D^b(X)$ of line bundles in the Thomsen collection of X. Conjecture 10.8 immediately follows. □

10.3. **The toric case of Orlov's Conjecture.** Theorem 10.1 also resolved the toric case of a conjecture of Orlov on the Rouquier dimension of derived categories. Let us begin with the definition of Rouquier dimension.

Definition 10.9 ([Rou08, Definition 3.2])**.** Let $\mathcal{T}$ be a triangulated category. If $T \in \mathcal{T}$ and $i \geq 0$, we let $\langle T \rangle_i$ denote the (not necessarily triangulated) subcategory of $\mathcal{T}$ given by direct summands of objects obtained by taking an i-fold extension of finite direct sums of shifts of T. The Rouquier dimension of $\mathcal{T}$ is defined to be $\dim \mathcal{T} := \inf\{d : \mathcal{T} = \langle T \rangle_{d+1} \text{ for some } T \in \mathcal{T}\}$.

Conjecture 10.10 ([Orl09b, Conjecture 10])**.** *If Y is a smooth, quasi-projective variety, then the equality $\dim D^b(Y) = \dim Y$ holds.*

See [BC23, §1.2] for known cases of Conjecture 10.10. This list predates the resolution of the toric case of Conjecture 10.10, which is due, independently, to Favero-Huang and Hanlon-Hicks-Lazarev:

Theorem 10.11 ([FH23, HHL24])**.** *Conjecture 10.10 holds when Y is a toric variety.*

Proof. This is immediate from [Rou08, Proposition 7.6] and Theorem 10.1, noting that Theorem 10.1 does not require X to be projective (see [HHL24, Corollary B]). □

11. Lecture 11 (Mahrud Sayrafi): The secondary fan

In this section we discuss the secondary fan of a multigraded ring S—a second fan associated to the toric varieties which arise from S via a Cox ring quotient construction like that in Theorem 4.6.

11.1. **Toric varieties of a multigraded ring.**

Distinct smooth projective toric varieties can have the same Cox ring with the same grading, adding a wrinkle to generalizations of the dictionary between projective space and the standard graded polynomial ring. The various possibilities are parametrized by what is called the secondary fan or GKZ fan of the toric variety. The goal of this subsection is to give a crash course in how to think about this fan, before we define it precisely.

Notation 11.1. We start with a polynomial ring $k[x_1, \dots, x_n]$ and a map $\deg\colon \mathbb{Z}^n \to \mathrm{Cl}(S)$ to a finitely generated abelian group $\mathrm{Cl}(S)$, which induces a $\mathrm{Cl}(S)$-grading on S that we also denote by deg. We will assume throughout that our grading satisfies the hypotheses of Proposition **??**, ensuring that there exists *some* simplicial projective toric variety with S as its Cox ring. Equivalently one may start with one such variety to obtain a suitable multigraded ring.

Let M be the kernel of deg, a lattice in $\mathbb{Z}^n$, and G the algebraic group $\operatorname{Spec} k[\mathrm{Cl}(S)]$. There is a short exact sequence

$$0 \to M \to \mathbb{Z}^n \to \mathrm{Cl}(S) \to 0. \tag{11.2}$$

Write $\nu_1, \dots, \nu_n$ for the rows of a matrix representing the map $M \to \mathbb{Z}^n$ and N for the dual lattice $\mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Z})$. (Note that we had fixed a basis of $\mathbb{Z}^n$ by choosing variables for S. The choice of ν_i also fixes a basis of M, for ease of representing fans in $N \otimes_{\mathbb{Z}} \mathbb{Q}$.)

The short exact sequence of vector spaces

$$0 \to M \otimes_{\mathbb{Z}} \mathbb{Q} \to \mathbb{Q}^n \to \mathrm{Cl}(S) \otimes_{\mathbb{Z}} \mathbb{Q} \to 0$$

has a dual sequence

$$0 \leftarrow N \otimes_{\mathbb{Z}} \mathbb{Q} \leftarrow \mathbb{Q}^n \leftarrow \mathrm{Hom}_{\mathbb{Q}}(\mathrm{Cl}(S) \otimes_{\mathbb{Z}} \mathbb{Q}, \mathbb{Q}) \leftarrow 0$$

which exchanges the role of the degrees $\deg x_i$ and the vectors ν_i. This relationship is known as Gale duality and can be used to develop the secondary fan. However, our setup has already broken the symmetry of degrees and rays by requiring that deg come from a *projective* toric variety, which forces the $\deg x_i$ to span a strongly convex cone while the ν_i span all of $N \otimes_{\mathbb{Z}} \mathbb{Q}$. (For more on this point see Appendix **??**, [CLS11, Proposition 14.3.2], and Remark 11.9.)

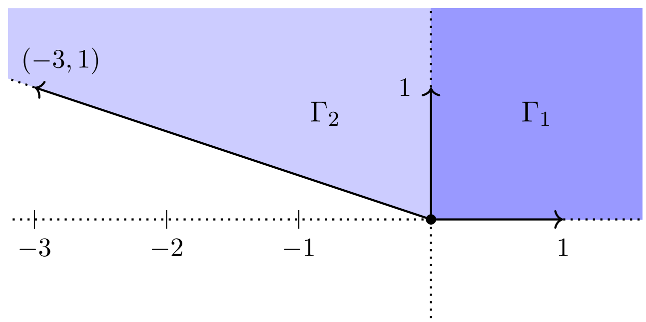
The crucial observation of the secondary fan is that there are only finitely many toric varieties coming from S, and in fact that they can be arranged into a polyhedral structure which reflects their geometry. The finitely many possibilities can be enumerated in several different ways, which we will address in turn through the subsequent subsections.

- **GIT quotient/ample divisor:** In Theorem 4.6 we saw that a simplicial toric variety is a geometric quotient of $\operatorname{Spec} S$. GIT quotients are parametrized by linearized line bundles. Choosing a linearization means choosing a divisor to be ample on the quotient, and we will see that the chambers of the secondary fan are in a sense possible nef cones.
- **polytope/normal fan:** If a projective toric variety has S as its multigraded Cox ring then we also know that its rays must come from (11.2). Up to translation, the polytopes with normal vectors in a fixed list lie in a fan. The cones of this "secondary" fan delineate possible normal fans of the polytopes.
- **irrelevant ideal:** Perhaps the simplest way to describe which toric varieties are possible is to list their irrelevant ideals, since S with its grading and a suitable irrelevant ideal fully determines a toric variety. Indeed the set of such ideals is easy to describe in terms of the map deg, and each ideal quickly determines both pieces of data above.

Some spaces appearing in the secondary fan are most properly interpreted as stacks, and some do not use all the rays from (11.2). We mostly avoid the former in these notes (see Remark 11.21) but address the latter in Section 11.5. Note that a variety with fewer than n rays will not actually have S as its own Cox ring, despite arising as a quotient of $\operatorname{Spec} S$.

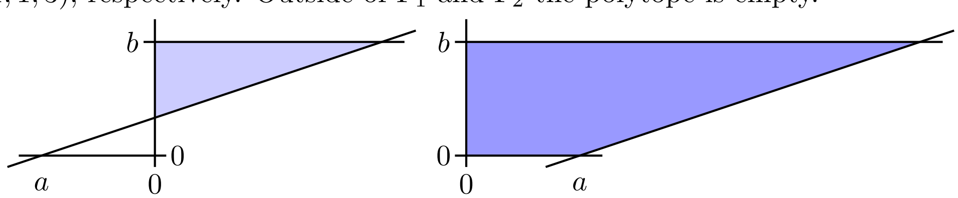
A reader primarily concerned with computation could start with Section 11.4 and take on faith that the ideals described there correspond to valid quotients and form a fan. Here we give an example of all three perspectives for our favorite toric variety. See also Theorem 11.30 for a summary.

Example 11.3. Consider $\mathcal{H}_3$, the Hirzebruch surface of type 3. Its Cox ring is the $\mathbb{Z}^2$-graded polynomial ring $S = k[x_0, x_1, x_2, x_3]$ so we take $\operatorname{Cl}(S) = \mathbb{Z}^2$. The degrees of the variables are $(1, 0), (-3, 1), (1, 0)$ and $(0, 1)$, respectively. These degrees partition the secondary fan into two chambers, which we label Γ_1 and Γ_2.



Each degree in $\operatorname{Cl}(S)$ gives a $(k^*)^2$-equivariant line bundle on $\operatorname{Spec} S$. Since degrees in the interior of Γ_1 are ample on $\mathcal{H}_3$, quotients in this region give back $\mathcal{H}_3$. Consider $d = (-1, 1)$, which lies in the interior of Γ_2. The subring of S consisting of elements whose degrees are multiples of d is isomorphic to the second Veronese subring of a polynomial ring with variables of degrees 1, 1, and 3. Proj of this ring is $\mathbb{P}(1, 1, 3)$. Notice that the quotient is a nonstandard description of $\mathbb{P}(1, 1, 3)$, whose own Cox ring has only 3 variables.

A degree $(a, b) \in \operatorname{Cl}(S)$ is in the image of the vector $(0, 0, a, b) \in \mathbb{Z}^4$. The polytope of this vector is defined by the restriction of the inequalities $x_0 \geq 0$, $x_1 \geq 0$, $x_2 \geq -a$, and $x_3 \geq -b$. In the coordinates y_0, y_1 we have chosen for M these give $y_0 \geq 0$, $y_1 \geq 0$, $-y_0 + 3y_1 \geq -a$, and $-y_1 \geq -b$. The inequalities bound a trapezoid for $(a, b) \in \Gamma_1$ and a triangle for $(a, b) \in \Gamma_2$, corresponding to $\mathcal{H}_3$ and $\mathbb{P}(1, 1, 3)$, respectively. Outside of Γ_1 and Γ_2 the polytope is empty.



To obtain irrelevant ideals for the chambers we look at which monomials have variables whose degrees span each cone. To span Γ_2 we must include x_1 and at least one other variable, giving the ideal $\langle x_0x_1, x_1x_2, x_1x_3\rangle$. Four subsets of two variables span Γ_1, giving $\langle x_0x_1, x_0x_3, x_1x_2, x_2x_3\rangle$.

11.2. **GIT quotients.** The sequence (11.2) induces maps of algebraic groups

$$0 \leftarrow \operatorname{Spec} k[M] \leftarrow (k^*)^n \leftarrow G \leftarrow 0. \tag{11.4}$$

The group $(k^*)^n$ acts on the affine space $\operatorname{Spec} S$ by componentwise multiplication so G does as well.

We will consider the quotient of $\operatorname{Spec} S$ by the action of G, which depends on a choice of linearized line bundle on $\operatorname{Spec} S$. We briefly recall some relevant terminology from geometric invariant theory (GIT). The case of torus actions on affine spaces is much simpler than the general theory, which can be found in [Mum65] or [Dol03].

Definition 11.5. A G-linearized line bundle on a space X with a G action is a line bundle $\mathcal{L}$ on X with an action of G compatible with the action on X such that the zero section of $\mathcal{L}$ is G-invariant. The G-equivariant Picard group of X is the group of G-linearized line bundles under tensor product, up to G-equivariant isomorphism.

Since $\operatorname{Spec} S$ is affine, it only has the trivial line bundle $\mathcal{O}$. Actions of G on $\mathcal{O}$ are given by elements of $\mathrm{Cl}(S)$.

Proposition 11.6. *In the setup of Notation 11.1 the grading group* $\mathrm{Cl}(S)$ *is the* G*-equivariant Picard group of* $\operatorname{Spec} S$*.*

Proof. An action of G on $\mathcal{O}$, the only isomorphism class of line bundle on $\operatorname{Spec} S$, is given by a character $\chi\colon G \to k^*$. The element $g \in G$ then acts on fibers of $\mathcal{O}$ by multiplication with $\chi(g)$. By definition of G the character χ is given by a map $\chi^\sharp\colon k[t^{\pm 1}] \to k[\mathrm{Cl}(S)]$ so that χ is a morphism of algebraic groups, and these are determined by the image $\chi^\sharp(t)$ in $\mathrm{Cl}(S)$ under the induced map of ordinary groups.

Tensor product of G-linearized line bundles corresponds to multiplication of characters and addition of elements in $\mathrm{Cl}(S)$. □

By Proposition 11.6 and [Mum65, Converse 1.13], the possible quotients of $\operatorname{Spec} S$ by G are parametrized by elements $d \in \mathrm{Cl}(S)$. Since $\operatorname{Spec} S$ had an action of $(k^*)^n$ the quotients retain an action of $\operatorname{Spec} k[M]$, a torus, via (11.4). In fact they are toric varieties, as we will see later by assigning them to fans.

According to GIT, in order to construct good (categorical) quotients we must first remove the unstable locus in $\operatorname{Spec} S$ corresponding to d. The quotient of the remaining semistable locus is then isomorphic to

$$\operatorname{Proj} \bigoplus_{i \in \mathbb{N}} \Gamma(\operatorname{Spec} S, \mathcal{L}_d^{\otimes i})^G$$

where $\mathcal{L}_d$ is the G-linearized line bundle of $d \in \mathrm{Cl}(S)$ and multiplication in the ring is given by tensor product of sections (see for instance [Dol03, Proposition 8.1]). In the toric case these invariant sections are elements of S.

Lemma 11.7. *The line bundle* $\mathcal{L}_d$ *on* $\operatorname{Spec} S$ *from Proposition 11.6 for* $d \in \mathrm{Cl}(S)$ *has* G*-invariant global sections* $\Gamma(\operatorname{Spec} S, \mathcal{L}_d)^G \cong S_d$*.*

Proof. The underlying line bundle of $\mathcal{L}_d$ is trivial so $\Gamma(\operatorname{Spec} S, \mathcal{L}_d) = S$. As in the proof of Proposition 11.6, an element g acts on $\mathcal{L}_d$ via multiplication by the image of g under the character χ corresponding to d. The $\mathbb{Z}^n$ in (11.2) is the character group of the torus and $S = k[\mathbb{Z}^n]$, so elements of the torus also act on monomials of S via the characters given by their exponents. Thus g acts on a monomial x^α with $\alpha \in \mathbb{Z}^n$ by $\chi(g^{-1})$ where χ is the character of $\deg \alpha$, so this monomial will be an invariant exactly when $\deg \alpha = d$. □

Write $S_{\mathbb{N}d}$ for the subring of S consisting of polynomials with degrees in the set $\mathbb{N}d$. By Lemma 11.7 and the discussion above it, these rings determine quotients of $\operatorname{Spec} S$.

Corollary 11.8. *The GIT quotients of* $\operatorname{Spec} S$ *by* G *are of the form* $X_d \coloneqq \operatorname{Proj} S_{\mathbb{N}d}$ *for* $d \in \mathrm{Cl}(S)$*.*

We will see more concrete constructions of the spaces X_d in the following subsections, along with descriptions of the unstable loci that were removed by the choice of line bundle $\mathcal{L}_d$. In particular, it is not obvious from the description in Corollary 11.8 that X_d is a toric variety. The secondary fan will additionally tell us which degrees d give the same unstable locus and thus the same quotient.

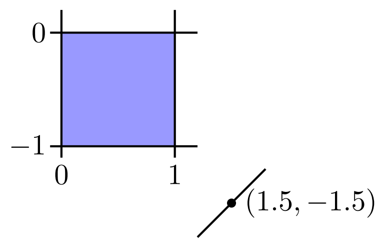
11.3. **Polytopes and the combinatorial secondary fan.** The toric structure of the quotients X_d connects their GIT problem to the combinatorial literature about the secondary fan. (See, for instance, [OP91].) In this context the secondary fan parametrizes polytopes whose normal vectors lie in a fixed list, up to translation. It will live in $\mathrm{Cl}(S) \otimes_{\mathbb{Z}} \mathbb{Q}$.

Write $\deg_{\mathbb{Q}}$ for the map $\mathbb{Q}^n \to \mathrm{Cl}(S) \otimes_{\mathbb{Z}} \mathbb{Q}$ induced by deg. Note that [CLS11, Proposition 4.2.6] implies that no variable x_i has $\deg_{\mathbb{Q}} x_i = 0$ even when $\mathrm{Cl}(S)$ contains torsion.

Remark 11.9. While Notation 11.1 sets up the case where deg is a positive grading and the quotients X_d are projective, there is also a secondary fan in the case where the vectors ν_i lie in an affine hyperplane, and thus the rays span a strongly convex cone. Instead of polytopes the chambers in the fan give triangulations on the hyperplane. This "GKZ fan" is explained in [GKZ94, Chapter 7].

Definition 11.10. Given $\alpha \in \mathbb{Q}^n$, write P_α for the polytope in $M \otimes_{\mathbb{Z}} \mathbb{Q}$ defined by the restriction of the inequalities $x_i \geq -\alpha_i$ for all i.

Example 11.11. Let $S = k[x_0, x_1, x_2, x_3, x_4]$ with degrees $(0,1,1)$, $(1,0,0)$, $(0,1,0)$, $(0,0,1)$, and $(1,1,0)$. Fixing a basis y_0, y_1 for M we can take $\nu_0 = (1,0)$, $\nu_1 = (0,1)$, $\nu_2 = (-1,1)$, $\nu_3 = (-1,0)$, $\nu_4 = (0,-1)$. Let $\alpha = (0,1,3,1,0)$. The inequality $x_2 \geq -3$ restricts to $(y_0, y_1) \cdot \nu_2 \geq -3$ or $-y_0 + y_1 \geq -3$ in $M \otimes_{\mathbb{Z}} \mathbb{Q}$. Combining with the other inequalities produces the polytope below.

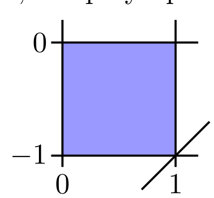


Note that the normal vectors of the polytope P_α lie in the list $\nu_1, \ldots, \nu_n$ from (11.2). However not all n inequalities/normal vectors need be required, if some of the hyperplanes $x_i = -\alpha_i$ do not intersect P_α in a facet (or at all, as in Example 11.11). Cox, Little, and Schenck refer to these as "virtual" facets. Replacing α with another vector of the same degree only translates the polytope.

Lemma 11.12. *If $\alpha, \alpha' \in \mathbb{Z}^d$ satisfy $\deg_{\mathbb{Q}} \alpha = \deg_{\mathbb{Q}} \alpha'$ then $P_{\alpha'} = m + P_\alpha$ for some $m \in M$.*

Hence P_α is determined by elements of $\mathrm{Cl}(S) \otimes_{\mathbb{Z}} \mathbb{Q}$ up to translation within $M \otimes_{\mathbb{Z}} \mathbb{Q}$. At certain degrees the face structure may change, when some hyperplanes intersect in unexpected codimension.

Example 11.13. Continuing Example 11.11, when $\alpha_1 + \alpha_3 = \alpha_2$ the hyperplane corresponding to x_2 intersects P_α in a vertex. For instance, the polytope below has $\alpha = (0,1,2,1,0)$.



This behavior only occurs on finitely many linear subspaces, creating polyhedral walls in the space of polytopes. The interiors of top dimensional cones collect all degrees in $\mathrm{Cl}(S) \otimes_{\mathbb{Z}} \mathbb{Q}$ whose polytopes share a normal fan. To describe lower dimensional cones we additionally specify for which indices i the hyperplanes $x_i = -\alpha_i$ do not intersect P_α, in order to distinguish cases like Examples 11.11 and 11.13. The two quotients share the fan for $\mathbb{P}^1 \times \mathbb{P}^1$, but we would record the set $\{2\}$ for Example 11.11 and $\varnothing$ for Example 11.13.

Definition 11.14. Let Σ be a generalized fan with support all of $N \otimes_{\mathbb{Z}} \mathbb{Q}$ and rays among the vectors $\nu_1, \dots, \nu_n$ from (11.2). Let F be a subset of the indices $1, \dots, n$ of ν_i *not* appearing in Σ.

Write $\tilde{\Gamma}(\Sigma, F)$ for the set of $\alpha \in \mathbb{Q}^n$ such that there exists a convex support function φ on Σ with $\varphi(\nu_i) = -\alpha_i$ for all $i \notin F$ and $\varphi(\nu_i) \geq -\alpha_i$ for all $i \in F$.

Write $\Gamma(\Sigma, F)$ for the image of $\tilde{\Gamma}(\Sigma, F) \subseteq \mathbb{Q}^n$ in $\mathrm{Cl}(S) \otimes_{\mathbb{Z}} \mathbb{Q}$ under the map induced by (11.2).

By [CLS11, Proposition 14.4.3] the set $\Gamma(\Sigma, F)$ is a strongly convex rational polyhedral cone. Compared to ordinary fans of toric varieties, generalized fans may contain cones which are not strongly convex (see [CLS11, Definition 6.2.2] and Remark 11.21). As with the behavior from Example 11.13, this does not occur for a general choice of α, and we will focus on the general (not generalized!) case.

Notably, a general polytope P_α will have full dimension and a simplicial normal fan. We now recall the definition and basic properties of this fan. For a vertex v of P_α take the cone of the set $P_\alpha \cap (M \otimes_{\mathbb{Z}} \mathbb{Q}) - v$. Over all vertices, the duals of these form the maximal cones of a complete fan in $N \otimes_{\mathbb{Z}} \mathbb{Q}$ by [CLS11, Proposition 14.2.10] (which also addresses lower dimensional P_α).

Rays of the normal fan correspond to facets of the polytope. In particular, the rays of the normal fan of P_α form a subset of $\{\nu_1, \dots, \nu_n\}$ by Definition 11.10. Nonsimplicial cones in the fan correspond to facets which intersect in unexpected codimension, as in Example 11.13.

Lemma 11.15. *Let $\alpha \in \mathbb{Q}^n$ such that P_α has full dimension and normal vectors exactly $\{\nu_i : i \notin F\}$ for $F \subseteq \{1, \dots, n\}$. Then $\alpha \in \tilde{\Gamma}(\Sigma, F)$ where Σ is the normal fan of P_α.*

Proof. We restrict our proof to the case when P_α is integral, in order to cite results which appear before the full definition of the normal fan in [CLS11, Proposition 14.2.10].

Let X_α be the toric variety of the polytope P_α. By [CLS11, Proposition 6.1.10] the divisor $\sum_{i \notin F} \alpha_i D_{\nu_i}$ is ample on X_α, so by [CLS11, Theorem 6.1.14] it has a strictly convex support function φ on Σ, satisfying $\varphi(\nu_i) = -\alpha_i$ for $i \notin F$. We claim that $\varphi(\nu_i) \geq -\alpha_i$ for $i \in F$.

Fix $i \in F$. Since Σ is the normal fan of a polytope it is complete, so we can write $\nu_i = \sum \beta_j \nu_j$ for $\beta_j > 0$ and ν_j the rays of a single cone in Σ. By the definition of a support function φ is linear within the cone spanned by the ν_j, so that $\varphi(\nu_i) = -\sum \beta_j \alpha_j$.

The rays of a cone in the normal fan correspond to hyperplanes intersecting in a face of P_α (swapping dimension and codimension), and this face consists of points m in P_α whose images $(\nu_j \cdot m)$ in $\mathbb{Q}^n$ satisfy $\nu_j \cdot m = -\alpha_j$. We also have $\nu_i \cdot m \geq -\alpha_i$, by definition of P_α. Hence

$$\varphi(\nu_i) = -\sum \beta_j \alpha_j = \sum \beta_j (\nu_j \cdot m) = \left(\sum \beta_j \nu_j\right) \cdot m = \nu_i \cdot m \geq -\alpha_i.$$

Notice that we get a strict inequality exactly when the hyperplane $x_i = -\alpha_i$ does not intersect P_α at all. The "general" case then corresponds to the interior of the cone $\tilde{\Gamma}(\Sigma, F)$. ☐

Remark 11.16. We have largely chosen to use polytopes instead of fans because we find the pictures of facets varying over the secondary fan to be more intuitive, and because we want to restrict the use of generalized fans and support functions to the discussion above. Lemma 11.15 connects the two descriptions. Compared to the fan Σ, the polytope P_α retains the data of a torus-invariant divisor on X_d. The existence of φ on the fan Σ requires its class $\deg \alpha$ to be nef on X_d, and the vector space structure on support functions clarifies the cone structure of $\Gamma(\Sigma, F)$.

Proposition 11.17 ([OP91, Corollary 3.6], [CLS11, Theorem 14.4.7])**.** *The collection of all cones $\Gamma(\Sigma, F)$ forms a rational polyhedral fan in $\mathrm{Cl}(S) \otimes_{\mathbb{Z}} \mathbb{Q}$, known as the* secondary fan *of the ring S graded via $\deg\colon \mathbb{Z}^n \to \mathrm{Cl}(S)$.*

The support of the secondary fan is the cone spanned by the $\deg_{\mathbb{Q}} x_i$, which in the setup of Notation 11.1 is strongly convex. We call maximal (here full dimensional) cones in the fan chambers*.*

Example 11.18. The secondary fan for the ring in Examples 11.11 and 11.13 is shown in Figure 11.1. The central cone contains the polytope whose normal fan defines a blowup X of $\mathbb{P}^1 \times \mathbb{P}^1$ at one torus-invariant point. This is the only type of polytope which requires all 5 inequalities.

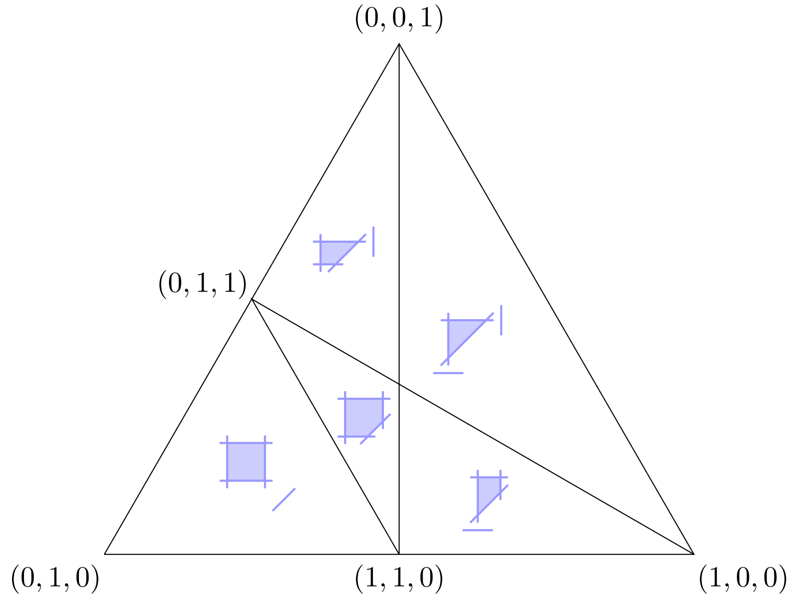


FIGURE 11.1. A cross-section of the secondary fan of $\mathrm{Bl}(\mathbb{P}^1 \times \mathbb{P}^1)$.

(Note that the `NormalToricVarieties` package for *Macaulay2* defaults to a basis for $\mathrm{Pic}(X)$ where the nef cone of X is the positive quadrant. For ease of visualization we have instead chosen the extremal rays of the secondary fan as our basis for $\mathrm{Cl}(S)$.)

The polytope from Example 11.11 is separated from the polytope for X by the wall corresponding to Example 11.13.

The toric variety X_d discussed in Section 11.2 is determined by the polytope P_α with $\deg \alpha = d$, meaning that it is the toric variety of the normal fan of P_α. Hence the secondary fan parametrizes the toric structures of the spaces X_d as the choice of d varies.

Theorem 11.19 ([CLS11, Theorem 14.2.13(b)])**.** *Suppose that $\alpha \in \mathbb{Z}^n$ and $\deg_{\mathbb{Q}} \alpha$ is in the interior of a chamber of the secondary fan of S. The toric variety of the polytope P_α is isomorphic to the quotient X_d of $\operatorname{Spec} S$ by G as given in Corollary 11.8 for $d = \deg \alpha$.*

The toric variety X_d is constant within the interior of each cone of the secondary fan. Varieties from distinct chambers will have distinct normal fans, though they may be isomorphic via a nontrivial map of lattices (as in Example 11.18, where the Hirzebruch surface $\mathcal{H}_1$ appears twice).

Corollary 11.20. *Suppose that $d, d' \in \mathrm{Cl}(S)$ are contained in the interior of the same chamber $\Gamma(\Sigma, F)$ of the secondary fan of S. Then $X_d \cong X_{d'}$ via the identity map on the fan Σ in $N \otimes_{\mathbb{Z}} \mathbb{Q}$.*

Proof. Choose $\alpha, \alpha' \in \mathbb{Z}^n$ so that $\deg \alpha = d$ and $\deg \alpha' = d'$. By definition of $\Gamma(\Sigma, F)$ we have $\alpha, \alpha' \in \tilde{\Gamma}(\Sigma, F)$. By Lemma 11.15 and Proposition 11.17 the normal fans of P_α and $P_{\alpha'}$ must both be Σ, since the interior of a chamber cannot intersect another cone in the same secondary fan. By Theorem 11.19 the varieties X_d and $X_{d'}$ are both defined by the fan Σ so they are isomorphic via the identity map. $\square$

Remark 11.21. Versions of Lemma 11.15, Theorem 11.19, and Corollary 11.20 are true for all $d \in \mathrm{Cl}(S)$, not only those in the interior of a chamber. Degrees on the walls of the secondary fan can have truly generalized fans, polytopes of dimension less than the rank of M, indices appearing neither among the rays of Σ nor in F (as in Example 11.13, which has $F = \varnothing$), and non-simplicial singularities which encourage us to consider them as stacks. See [BEF26] for the relevant constructions in these more technical cases.

11.4. **Irrelevant ideals.** We now describe the secondary fan more concretely in terms of the function $\deg\colon \mathbb{Z}^n \to \mathrm{Cl}(S)$. Its cones correspond to possible irrelevant ideals for the varieties X_d, and the list of possible ideals can be computed without any knowledge of GIT or generalized fans and support functions.

Theorem 11.22. *The secondary fan of S graded via $\deg\colon \mathbb{Z}^n \to \mathrm{Cl}(S)$ is the coarsest common refinement of the vectors $\deg_{\mathbb{Q}} x_i$: it refines all fans with this list of rays, and it coarsens all other common refinements.*

We postpone the proof of Theorem 11.22 until the end of the section. The idea is that the chamber of the secondary fan containing a degree $d \in \mathrm{Cl}(S) \otimes_{\mathbb{Z}} \mathbb{Q}$ is fully determined by the subsets of variables whose degrees span a cone containing d.

Definition 11.23. Fix a monomial $x^\alpha \coloneqq x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ and a degree $d \in \mathrm{Cl}(S) \otimes_{\mathbb{Z}} \mathbb{Q}$. For a strongly convex cone $\Gamma \subseteq \mathrm{Cl}(S) \otimes_{\mathbb{Z}} \mathbb{Q}$, let

$$\mathrm{ideal}(\Gamma) \coloneqq \langle x^\alpha \in S \mid x^\alpha \text{ spans } d \text{ for all } d \in \Gamma \rangle.$$

Conversely, given a monomial ideal B, let

$$\mathrm{cone}(B) \coloneqq \{ d \in \mathrm{Cl}(S) \otimes_{\mathbb{Z}} \mathbb{Q} \mid x^\alpha \text{ spans } d \text{ for all } x^\alpha \in B \}.$$

These operations are inverses of each other on the secondary fan, and define an inclusion-reversing correspondence between cones and irrelevant ideals. To mimic the ordinary construction of a toric variety from $\operatorname{Spec} S$, we should also describe $\mathrm{ideal}(\Gamma(\Sigma, F))$ in terms of the fan Σ.

Proposition 11.24. *Suppose $\Gamma = \Gamma(\Sigma, F)$ is a chamber of the secondary fan of S. The minimal generators of* $\mathrm{ideal}(\Gamma)$ *are the monomials $\prod_{\nu_i \notin \sigma(1)} x_i$ for σ a maximal cone of Σ. Further we have* $\mathrm{cone}(\mathrm{ideal}(\Gamma)) = \Gamma$.

Note that σ may contain ν_i for $i \in F$ but these are not rays of Σ and thus are not contained in $\sigma(1)$. In fact, for $i \in F$ the variable x_i will appear in every generator of $\mathrm{ideal}(\Gamma)$ (see Lemma 11.28).

Proof. This is almost a restatement of [CLS11, Proposition 15.2.1], after untangling some notation. Recall from Lemma 11.15 and the discussion before it that Σ is a simplicial fan. Let x^α be a monomial of the form $\prod_{\nu_i \notin \sigma(1)} x_i$ for σ a maximal cone of Σ.

Since σ is simplicial, [CLS11, Proposition 14.3.1] implies that the monomial x^α comes from a "β-basis," so it spans Γ by [CLS11, Proposition 15.2.1(b)]. A subset of the variables spans a cone of smaller dimension, which cannot contain a chamber, so x^α is a minimal generator of $\mathrm{ideal}(\Gamma)$.

Immediately from Definition 11.23 we have $\Gamma \subseteq \mathrm{cone}(\mathrm{ideal}(\Gamma))$. By [CLS11, Proposition 15.2.1(c)] the cone Γ is the intersection of the spans of monomials $\prod_{\nu_i \notin \sigma(1)} x_i$. This will imply that $\Gamma = \mathrm{cone}(\mathrm{ideal}(\Gamma))$ once we show that all minimal generators of $\mathrm{ideal}(\Gamma)$ have the correct form, since multiplying by additional variables only increases the span of a monomial.

Let x^α be a minimal generator of $\mathrm{ideal}(\Gamma)$. We claim that the subset of nonzero $\deg_{\mathbb{Q}} x_i^{\alpha_i}$ is linearly independent. They span a strongly convex cone by the assumptions of Notation 11.1 (see Lemma **??**). This cone has a simplicial subdivision (e.g. by [OP91, Corollary 3.8]), and by the discussion before [CLS11, Proposition 15.2.10] the chamber Γ is contained in a single cone. Since x^α is a minimal generator of $\mathrm{ideal}(\Gamma)$ no proper factor can span Γ, so x^α itself spans a simplicial cone. The existence of σ then follows from [CLS11, Proposition 15.2.1(b)]. ∎

As in Theorem 4.6, the vanishing locus of the irrelevant ideal should be removed to form the quotient X_d. In the language of Section 11.2, it defines the unstable locus. Additionally, when d is in the interior of a chamber the stable and semistable loci agree, meaning the quotient is *geometric*.

Theorem 11.25. *Suppose $d \in \mathrm{Cl}(S)$ is contained in the interior of a chamber Γ of the secondary fan of S. The unstable locus of the line bundle $\mathcal{L}_d$ on $\operatorname{Spec} S$, as in Proposition 11.6, is $V(\mathrm{ideal}(\Gamma))$. The space X_d, as in Corollary 11.8, is a simplicial projective toric variety and a geometric quotient of $\operatorname{Spec} S \setminus V(\mathrm{ideal}(\Gamma))$ by G.*

Proof. Write $\Gamma = \Gamma(\Sigma, F)$. Then [CLS11, Corollary 14.2.22] describes the semistable locus as the complement of $V(Z(\chi))$ for an ideal $Z(\chi)$ defined in terms of the "virtual facets" of P_α for $\deg \alpha = d$. By their equation (14.4.11) it can also be described in terms of vertices of P_α, or equivalently maximal cones of Σ, giving $\mathrm{ideal}(\Gamma)$ as described in Proposition 11.24.

The fan Σ is simplicial and complete by the discussion before Lemma 11.15, and X_d is the toric variety of Σ by Theorem 11.19. Therefore X_d is simplicial, a geometric quotient by Theorem 4.6, and projective because it is the toric variety of a full dimensional rational polytope. ☐

Proof of Theorem 11.22 cf. [ADHL15, Theorem 2.2.2.3]. Fix d in the interior of a chamber Γ of the secondary fan. We must show that Γ is the intersection over all fans Δ on the rays $\deg_{\mathbb{Q}} x_i$ of the cone in Δ which contains d. By the proof of Proposition 11.24 it is the intersection of simplicial cones on the rays $\deg_{\mathbb{Q}} x_i$. Thus we must show that these intersections are the same.

As in the proof of Proposition 11.24, a cone containing d from an arbitrary fan Δ has a simplicial cone on a subset of its rays that also contains d. Conversely, an arbitrary simplicial cone can be extended to a fan Δ on the remaining rays by a variant of the construction from [CLS11, Lemma 15.2.10] (altered for the case where Δ is not complete). ☐

11.5. **Nef and moving cones.** In either of the constructions of X_d, the resulting torus-invariant divisor must be ample. From the GIT perspective we are using the line bundle $\mathcal{L}_d$ to embed X_d in projective space, and from the toric perspective the divisor of the polytope P_α is ample on the variety of its normal fan. However, we must be careful with what we mean by the divisor corresponding to d in the case where the class group of X_d is smaller than $\mathrm{Cl}(S)$.

Lemma 11.26. *Suppose $d \in \mathrm{Cl}(S)$ is in the interior of a chamber of the secondary fan. There is an induced projection $\mathrm{Cl}(S) \to \mathrm{Cl}(X_d)$.*

Proof. Let Σ be the fan of X_d. By Definition 11.14 and Theorem 11.19 the rays in $\Sigma(1)$ correspond to a subset of the vectors $\{\nu_1, \dots, \nu_n\}$, giving an injection $\mathbb{Z}^{\Sigma(1)} \to \mathbb{Z}^n$ which commutes with the maps of rays:

$$\begin{array}{ccc} N & \longleftarrow & \mathbb{Z}^n \\ \uparrow & & \uparrow \\ N & \longleftarrow & \mathbb{Z}^{\Sigma(1)}. \end{array}$$

Taking $\mathrm{Hom}_{\mathbb{Z}}(-, \mathbb{Z})$ and using (4.3) we get

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M & \longrightarrow & \mathbb{Z}^n & \longrightarrow & \mathrm{Cl}(S) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & & & \\ 0 & \longrightarrow & M & \longrightarrow & \mathbb{Z}^{\Sigma(1)} & \longrightarrow & \mathrm{Cl}(X_d) & \longrightarrow & 0 \end{array}$$

where the first map is the identity on M and the second is surjective. This induces a surjection $\mathrm{Cl}(S) \to \mathrm{Cl}(X_d)$. ☐

Using the projection of class groups, we can give a more precise description of the ample and nef classes on X_d. If $\mathrm{Cl}(S)$ is the class group of X_d then the nef cone will be exactly the cone containing (the rational image of) d in the secondary fan.

Theorem 11.27 ([CLS11, Theorem 15.1.3(a)]). *Suppose $d \in \mathrm{Cl}(S)$ is in the interior of a chamber Γ of the secondary fan of S. The cone* $\mathrm{Nef}\, X_d$ *is the projection of Γ to $\mathrm{Cl}(X_d) \otimes_{\mathbb{Z}} \mathbb{Q}$ via the map from Lemma 11.26.*

A variety X_d with fewer than n rays still has a nef cone related to its chamber Γ, and still arises as a GIT quotient of $\mathrm{Spec}\, S$, but the short exact sequence computing its class group no longer exactly matches (11.2).

Similarly, the Cox ring of such an X_d is not actually S, as a polynomial ring with fewer variables is sufficient to construct the toric variety. Thus in one sense $\mathrm{ideal}(\Gamma)$ is an irrelevant ideal—it defines a permissible unstable locus in $\mathrm{Spec}\, S$—yet it is not the irrelevant ideal of X_d in the ordinary sense.

Lemma 11.28. *The following are equivalent for d in the interior of a chamber $\Gamma(\Sigma, F)$ of the secondary fan:*

(1) The Cox ring of X_d is S;
(2) $\mathrm{Cl}(X_d) = \mathrm{Cl}(S)$;
(3) The irrelevant ideal of X_d is $\mathrm{ideal}(\Gamma(\Sigma, F))$;
(4) $\mathrm{ideal}(\Gamma(\Sigma, F))$ is not contained in a principal ideal $\langle x_i \rangle$;
(5) The rays of Σ are precisely $\nu_1, \dots, \nu_n$;
(6) $F = \varnothing$; and
(7) d is in the moving cone *of $\mathrm{Cl}(S)$.*

We saw in Section 5.3 that if X_d satisfies the conditions of Lemma 11.28, a (finitely generated) $\mathrm{Cl}(S)$-graded S-module M induces a (coherent) sheaf on X_d. To emphasize the dependence on X_d, we write $M|_{X_d}$ for this sheaf. A minor modification of the associated sheaf construction works for the other X_ds as well.

Proposition 11.29. *Suppose d is in the interior of a chamber of the secondary fan of S and $\mathcal{E}$ is a quasi-coherent sheaf on X_d. Then there exists a $\mathrm{Cl}(S)$-graded S-module M such that $M|_{X_d} \cong \mathcal{E}$.*

Proof. This generalizes Proposition 5.7 without assuming the circumstances of Lemma 11.28. We only sketch the additional argument. See also [Cox95, Theorem 3.2] and [BEF26, Corollary 3.6].

Write $\Gamma = \Gamma(\Sigma, F)$ for the chamber containing d. Generators x^α of $\mathrm{ideal}(\Gamma)$ have the form $\prod_{\nu_i \notin \sigma} x_i$ for σ a cone of Σ by Proposition 11.24. The sheaf $M|_{X_d}$ is defined by the degree $0 \in \mathrm{Cl}(S)$ part of the localization M_{x^α} on the open set of X_d where x^α doesn't vanish. Recall that x_i divides all generators of $\mathrm{ideal}(\Gamma)$ when $i \in F$, so that x_i and x_i^{-1} act isomorphically on M_{x^α} for each x^α.

Write $\varphi\colon \mathrm{Cl}(S) \to \mathrm{Cl}(X_d)$ for the projection from Lemma 11.26. Given a quasi-coherent sheaf $\mathcal{E}$ on X_d, [Cox95, Theorem 3.2] constructs a $\mathrm{Cl}(X_d)$-graded module N over $k[x_i \mid i \notin F]$ such that $\widetilde{M} = \mathcal{E}$ in the ordinary sense. To extend N to S and $\mathrm{Cl}(S)$ define $M_d = N_{\varphi(d)}$ for each $d \in \mathrm{Cl}(S)$ and let x_i act on M by the identity on N for all $i \in F$. $\square$

11.6. **Summary.** To finish, we collect statements from previous subsections for reference.

Theorem 11.30. *Let S and $\deg\colon \mathbb{Z}^n \to \mathrm{Cl}(S)$ be as in Notation 11.1. The secondary fan of S is the coarsest common refinement of the vectors $\deg_{\mathbb{Q}} x_i$. Its cones are indexed by fans Σ and indices $F \subseteq \{1, \dots, n\}$. Suppose $d \in \mathrm{Cl}(S)$ lies in the interior of a chamber $\Gamma = \Gamma(\Sigma, F)$ and $B = \mathrm{ideal}(\Gamma)$. Choose $\alpha \in \mathbb{Z}^n$ with $\deg \alpha = d$.*

(1) The degree d gives a G-linearized line bundle $\mathcal{L}_d$ on $\mathrm{Spec}\, S$. The unstable locus of $\mathcal{L}_d$ is $V(B)$ and the geometric quotient $X_d = \mathrm{Proj}\, S_{\mathbb{N}d}$ is a simplicial projective toric variety.
(2) The simplicial fan Σ is the normal fan of P_α and defines X_d. Its rays are a subset of the vectors $\nu_1, \dots, \nu_n$ from (11.2). For $i \notin F$ the hyperplane corresponding to ν_i cuts out a facet of P_α and for $i \in F$ it does not intersect P_α.
(3) The group $\mathrm{Cl}(X_d)$ is a quotient of $\mathrm{Cl}(S)$, and they are equal under the conditions from Lemma 11.28, in which case $\Gamma = \mathrm{Nef}(X_d)$.

Proof. The secondary fan is the coarsest common refinement by Theorem 11.22. Its cones are constructed in Definition 11.14.

The G-linearized line bundle $\mathcal{L}_d$ is described in Proposition 11.6. The unstable locus of $\mathcal{L}_d$ is $V(B)$, the quotient is geometric, and the resulting space is a simplicial projective toric variety by Theorem 11.25. It is isomorphic to $\operatorname{Proj} S_{\mathbb{N}d}$ by Corollary 11.8.

Let Σ' be the normal fan of P_α and let F' index the rays not appearing in Σ'. Then $\alpha \in \tilde{\Gamma}(\Sigma', F')$ by Lemma 11.15 so $d \in \Gamma(\Sigma', F')$. Since d is in the interior of $\Gamma(\Sigma, F)$ by assumption and these cones form a fan by Theorem 11.17, we must have $\Sigma' = \Sigma$ and $F' = F$.

Therefore X_d is the toric variety of Σ by Theorem 11.19. The fan Σ is simplicial and has rays among the ν_i by the discussion beforehand. For $i \in F$ the hyperplane corresponding to ν_i intersects P_α in a facet because Σ is the normal fan of P_α. For $i \notin F$ the hyperplane does not intersect P_α by the proof of Lemma 11.15 because α lies in the interior of $\tilde{\Gamma}(\Sigma, F)$.

The group $\mathrm{Cl}(X_d)$ is a quotient of $\mathrm{Cl}(S)$ by Lemma 11.26. If they are equal then $\Gamma = \mathrm{Nef}(X_d)$ by Theorem 11.27. ∎

12. Lecture 12 (Martin Kalck); The Cox Category, I: Definition

As we saw in Theorem 2.5, the Beilinson collection of line bundles $\mathcal{O}_{\mathbb{P}^n}(-n), \dots, \mathcal{O}_{\mathbb{P}^n}(-1), \mathcal{O}_{\mathbb{P}^n}$ is a full strong exceptional collection for $D^b(\mathbb{P}^n)$. This result extends to products of projective spaces.

Example 12.1. The Beilinson collection for $\mathbb{P}^1$ consists of the line bundles of degrees -1 and 0. On $\mathbb{P}^1 \times \mathbb{P}^1$, the bundles $\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(-1,-1), \mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(-1,0), \mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(0,-1), \mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}$ form a full, strong exceptional collection. Notice that the line bundles that arise are the box products of the bundles in the Beilinson collections of the factors. A similar result holds for any product $\mathbb{P}^n \times \mathbb{P}^m$. In this case, the line bundles that appear are $\mathcal{O}(-i,-j)$ for $-n \le -i \le 0$ and $-m \le -j \le 0$. In fact, this extends to any product $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_t}$. The proof follows easily from Beilinson's original work. This is for the following reasons:

1. Exceptionality is a simple sheaf cohomology calculation.
2. Fullness follows easily from having a resolution of the diagonal like Beilinson's.
3. If F is Beilinson's resolution of the diagonal for $\mathbb{P}^n$ and G is Beilinson's resolution of the diagonal for $\mathbb{P}^m$, then $F \boxtimes G$ is a resolution of the diagonal for $\mathbb{P}^n \times \mathbb{P}^m$.

For instance, let us consider the case of $\mathbb{P}^1 \times \mathbb{P}^1$. The fact that

$$\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(-1,-1), \mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(-1,0), \mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(0,-1), \mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}$$

form a strong exceptional collection is a simple cohomology computation, using the Künneth formula. For fullness, we start with the fact that

$$\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1} \leftarrow \mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(-1,-1)$$

is a resolution of $\Delta_{\mathbb{P}^1} \subseteq \mathbb{P}^1 \times \mathbb{P}^1$. Taking the box product of this complex with itself yields a resolution of $\Delta_{\mathbb{P}^1\times\mathbb{P}^1} \subseteq (\mathbb{P}^1)^4$:

$$\mathcal{O}_{(\mathbb{P}^1)^4} \leftarrow \begin{matrix} \mathcal{O}_{(\mathbb{P}^1)^4}(-1,0,-1,0) \\ \oplus \\ \mathcal{O}_{(\mathbb{P}^1)^4}(0,-1,0,-1) \end{matrix} \leftarrow \mathcal{O}_{(\mathbb{P}^1)^4}(-1,-1,-1,-1).$$

Applying Construction 10.5 gives fullness, just as in the proof of Theorem 2.5.

Some other classical examples of smooth projective toric varieties also satisfy an analogue of Beilinson's theorem.

Example 12.2. If X is a Hirzebruch surface, then

$$\mathcal{O}_X(-1,-1), \mathcal{O}_X(-1,0), \mathcal{O}_X(0,-1), \mathcal{O}_X$$

is a full, strong exceptional collection for $D^b(X)$. For those who are curious: this is a special case of Orlov's semiorthogonal decomposition result for projective bundles [Huy06, Cor. 8.36].

This raises a natural question: which smooth, projective toric varieties satisfy an analogue of Beilinson's theorem? In 1997, King conjectured that the answer was "all of them".

Conjecture 12.3 (King, 1997)**.** *Every smooth, projective toric variety has a full, strong exceptional collection of line bundles.*

Remark 12.4. A variant of Corollary 2.6 holds for any variety satisfying King's Conjecture.

King's Conjecture turned out to be false. Hille-Perling gave the first counterexamples, and followup work of Michalek and Efimov gave counterexamples for fairly simple varieties and showed that the conjecture cannot be salvaged by restricting, for instance, to Fano toric varieties [HP06, Mic11, Efi14].

Nevertheless, King's Conjecture continued to inspire work on exceptional collections for toric varieties, and there is a huge literature in this vein: [ABKW20, BFK19, BDLM20, BIL+26, BDM22, BT09,BH09,BO19,BW21,CMR04,CMR10,DLM09,Jer15,Jer17,Jer20,Kaw06,Kaw13,LM11,OU13, PN17,San23,Ueh14], *inter alia*. The closest result to King's Conjecture was probably Kawamata's theorem, which proved a weakening of the conjecture by dropping the line bundle requirement and removing the strongness condition [Kaw06]. While this is a major result, the contrast with Beilinson's theorem is pretty stark: while Corollary 2.6 yields unique representations of any sheaf in terms of well-known building blocks, namely the line bundles in the Beilinson collection, the representations from Kawamata's would fail to be unique, and they would be in terms of building blocks that we don't understand whatsoever. Notably: it is an open question whether the objects in Kawamata's collection are sheaves or whether they will necessarily involve complexes of sheaves.

Over the remaining lectures, we will develop and prove an alternative realization of King's Conjecture. (The king is dead, long live the king!) The basic idea originates from three observations:

(1) Several distinct toric varieties can have the same Cox ring.
(2) The Hanlon-Hicks-Lazarev Theorem shows that the analogue of Beilinson's resolution of the diagonal is uniform for all toric varieties with the same Cox ring.
(3) The Bondal-Thomsen collection–which is also uniform for all toric varieties with the same Cox ring–is an excellent analogue of the Beilinson collection in many ways.

The picture that emerges is the following:

$\mathbb{P}^n$	Smooth, projective toric variety
Beilinson collection	Bondal-Thomsen collection
Beilinson resolution of diagonal	HHL resolution of diagonal
Theorem 2.5 (King's Conjecture for $\mathbb{P}^n$)	**Revised King's Conjecture**

Note that the Bondal-Thomsen collection and the HHL resolution depend only on the Cox ring. This suggests that the "right" analogue of Theorem 2.5, i.e. the "right" version of King's Conjecture, should perhaps depend only on the Cox ring. That is, a revised King's Conjecture should involve a category that combines the sheaves from all of the toric varieties with the same Cox ring.

This only raises another question: if X and X' are two toric varieties with the same Cox ring, how do "combine" their derived categories?

More specifically, we have seen that a multigraded polynomial ring S determines a finite set of simplicial toric varieties $X_1, X_2, \dots, X_r$ each of which has a corresponding stack $\mathcal{X}_1, \mathcal{X}_2, \dots, \mathcal{X}_r$. How can we combine their derived categories? That is to say: we want to define a single category–which we refer to as the Cox category, and denote by $D_{\mathrm{Cox}}(X)$ or $D_{\mathrm{Cox}}(S)$–that combines the study of modules on these $\mathcal{X}_i$ *simultaneously*. The theorem we are driving for is:

Theorem 12.5 (Revised King's Conjecture)**.** *If X is a projective toric variety with Cox ring S, then the Bondal–Thomsen collection, under a natural ordering, forms a full strong exceptional collection of line bundles for $D_{\mathrm{Cox}}(S)$.*

This should be viewed as a vague statement for the moment, as we have not defined $D_{\mathrm{Cox}}(S)$ precisely, nor have we have identified the Bondal–Thomsen collection as a subset of the Cox category. But it gives a sense of where we are going: for toric varieties, the Bondal–Thomsen collection and the resolution of the diagonal only depend on the Cox ring. And Theorem 12.5 is a revised King's Conjecture, and an extension of Theorem 2.5, that also only depends on the Cox ring. We now aim to make the theorem precise.

Lemma 12.6. *There exists a smooth, toric stack $\widetilde{\mathcal{X}}$ that admits proper birational maps $\pi_i \colon \widetilde{\mathcal{X}} \to \mathcal{X}_i$ for all $i = 1, 2, \dots, r$.*

Proof. At the level of varieties, this would be very standard as one could just take a simplicial refinement of the fans of each of the X_i. Extending that proof to toric stacks requires working with additional notation, but there are very few new ideas and so we omit the proof. See [BBB+24, §3.1] for details. □

A standard fact [BBB+24, Lemma 3.7] is that the (derived) pullback π_i^* from $D(\mathcal{X}_i)$ to $D(\widetilde{\mathcal{X}})$ is fully faithful.[7] Thus, π_i^* embeds each derived category of $\mathcal{X}_i$ into the derived category of $\widetilde{\mathcal{X}}$. This provides a way to "combine" $D(\mathcal{X}_1), D(\mathcal{X}_2), \cdots, D(\mathcal{X}_r)$: we can take the span of these categories inside of the ambient category $D(\widetilde{\mathcal{X}})$.

Definition 12.7. We define the Cox category of S, denoted $D_{\text{Cox}}(S)$ as follows. Since each pullback functor π_i^* is fully faithful, $\pi_i^* D(\mathcal{X}_i)$ is a copy of $D(\mathcal{X}_i)$ that lives in $D(\widetilde{\mathcal{X}})$. We define $D_{\text{Cox}}(S)$ as the subcategory of $D(\widetilde{\mathcal{X}})$ generated[8] by $\pi_1^* D(\mathcal{X}_1), \pi_2^* D(\mathcal{X}_2), \ldots, \pi_r^* D(\mathcal{X}_r)$. In other words, $D_{\text{Cox}}(S)$ is like the category generated by (the pullbacks of) all sheaves from all of the $\mathcal{X}_i$.

It is easy to check that the definition of $D_{\text{Cox}}(S)$ does not depend upon the choice of $\widetilde{\mathcal{X}}$ from Lemma 12.6; see [BBB+24, Corollary 3.9]. But these copies overlap.

Example 12.8. For any $1 \leq i \leq r$ consider the structure sheaf $\mathcal{O}_{\mathcal{X}_i}$. We have $\pi_i^* \mathcal{O}_{\mathcal{X}_i} = \mathcal{O}_{\widetilde{\mathcal{X}}}$. Thus $\mathcal{O}_{\widetilde{\mathcal{X}}}$ lies in the intersection of the copy of each $D(\mathcal{X}_i)$.

We can thus think of $D_{\text{Cox}}(S)$ as "gluing" the various $D(\mathcal{X}_i)$ together. See Figure 12.1. The main results of the next several lectures show that $D_{\text{Cox}}(S)$ has nicer properties than any of the individual $D(\mathcal{X}_i)$, in a manner that is (metaphorically at least) akin to how a projective variety can have nicer properties than its various affine patches.

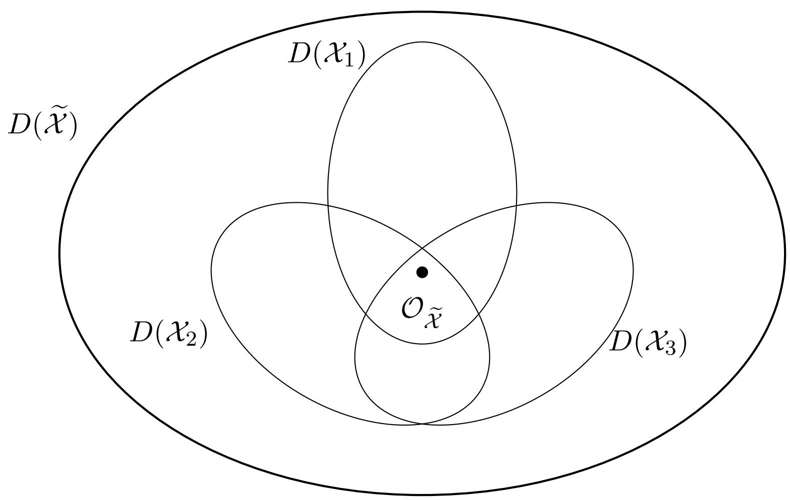


FIGURE 12.1. The Cox category is built by embedding the categories of the $\mathcal{X}_i$ into the ambient category $D(\widetilde{\mathcal{X}})$. These copies overlap, and so the Cox category is like a gluing of the sheaves on the $\mathcal{X}_i$.

Example 12.9. Let S be the Cox ring of $\mathbb{P}^n$. In this case, the secondary fan only has a single chamber, i.e. the chamber corresponding to $\mathbb{P}^n$ and thus $D_{\text{Cox}}(S) = D(\mathbb{P}^n)$. The same holds for any toric variety whose secondary fan has a single chamber, such as weighted projective space or a product of projective spaces.

[7] Because we are focused on derived categories, every time we write π_i^* we should really be using the derived functor $L\pi_i^*$. However, since we focus almost entirely on the pullbacks of line bundles, $L\pi_i^*$ and π_i^* will coincide. We will thus abuse notation and write π_i^* throughout.

[8] By "generated" we mean the smallest triangulated category containing these objects and that is closed under taking summands.

Example 12.10. Let S be the Cox ring of $\mathcal{H}_1$ the Hirzebruch surface of type 1. In this case, the secondary fan has two chambers: one chamber corresponds to $\mathbb{P}^2$ and one corresponding to $\mathcal{H}_1$. There is a natural morphism $\alpha\colon \mathcal{H}_1 \to \mathbb{P}^2$ given by the blowup of $\mathbb{P}^2$ at point, and thus we can take $\widetilde{X} = \mathcal{H}_1$, $\pi_1 = \alpha\colon \mathcal{H}_1 \to \mathbb{P}^2$, and $\pi_2 = \mathrm{id}\colon \mathcal{H}_1 \to \mathcal{H}_1$. We thus have $D_{\mathrm{Cox}}(S) = \langle \pi_1^* D(\mathbb{P}^2), \pi_2^* D(\mathcal{H}_1)\rangle = D(\mathcal{H}_1)$.

Example 12.11. Let S be the Cox ring of $\mathcal{H}_3$ the Hirzebruch surface of type 3. In this case, the secondary fan has two chambers: one chamber corresponds to a weighted projective space $\mathbb{P}(1,1,3)$ and one corresponding to $\mathcal{H}_3$. Unlike the previous example, $\mathbb{P}(1,1,3)$ is simplicial but not smooth, and so we need to consider the weighted projective stack $\mathbb{P}_{\mathrm{stack}}(1,1,3)$. There is a morphism from $\mathcal{H}_3$ to the weighted projective space $\mathbb{P}(1,1,3)$, where the pullback of the line bundle $\mathcal{O}_{\mathbb{P}(1,1,3)}(3)$ is $\mathcal{O}_{\mathcal{H}_3}(0,1)$. But there is no such morphism to $\mathbb{P}_{\mathrm{stack}}(1,1,3)$. The reason for this is simple: if there were such a morphism $\alpha\colon \mathcal{H}_3 \to \mathbb{P}_{\mathrm{stack}}(1,1,3)$, then since $\mathcal{O}(1)$ is a line bundle on the stack, its pullback would need to be a line L such that $L^{\otimes 3} = \alpha^*\mathcal{O}(3) = \mathcal{O}_{\mathcal{H}_3}(0,1)$, which is impossible. This issue is avoided on the weighted projective space because $\mathcal{O}(1)$ is not a line bundle on that space. Thus, we have our first example where $D_{\mathrm{Cox}}(S)$ is a genuine combination of two subcategories.

Remark 12.12. It is rarely the case that there is some $\widetilde{\mathcal{X}}$ as in Lemma 12.6 where $D_{\mathrm{Cox}}(S)$ equals $D(\widetilde{\mathcal{X}})$. In other words, $D_{\mathrm{Cox}}(S)$ is generally a proper subcategory of $D(\widetilde{\mathcal{X}})$, as illustrated somewhat cartoonishly in Figure 12.1. Nevertheless, it can sometimes be helpful to think of the Cox category as the derived category of a mythical toric variety $\mathcal{X}_{\mathrm{myth}}$ associated to the Cox ring S. This $\mathcal{X}_{\mathrm{myth}}$ would be smooth and proper and birational to each of the $\mathcal{X}_i$. This yields a heuristic for properties of $D_{\mathrm{Cox}}(S)$, and as we will see in §17.1, some of those resulting predictions are correct.

Remark 12.13. Let us continue with the metaphor that $D_{\mathrm{Cox}}(S)$ is like a projective variety, obtained by gluing the "affine patches" $D(\mathcal{X}_i)$. One might ask: what is the analogue of the transition functions from one affine patch to another? The answer is the Fourier–Mukai transforms associated to the birational maps $\mathcal{X}_i \dashrightarrow \mathcal{X}_j$. We will examine these transforms in greater depth in Section 13. One can also make this metaphor more precise, and totally excise the cover $\widetilde{\mathcal{X}}$, by using the Grothendieck construction [Gro63, §VI.8]. This gets pretty technical, and so we omit it in these notes; see [BBB+24, Appendix A] for details.

We have defined $D_{\mathrm{Cox}}(S)$, but to make Theorem 12.5 precise, we also need to define the Bondal–Thomsen collection as a subset of $D_{\mathrm{Cox}}(S)$. We return to our running example before giving the full definition.

Example 12.14. Let $X = \mathcal{H}_3$ be a Hirzebruch surface of type 3. As we have already seen, its secondary fan has two maximal chambers (see Figure 12.2): one corresponding to $\mathcal{H}_3$, and the other to the weighted projective stack $\mathbb{P}(1,1,3)$. Thus, D_{Cox} is the subcategory of $D(\widetilde{\mathcal{X}})$ generated by pullbacks of line bundles from $\mathcal{H}_3$ and $\mathbb{P}(1,1,3)$. Some of these "glue," for example, the pullbacks of $\mathcal{O}_{\mathcal{H}_3}(0,d)$ and $\mathcal{O}_{\mathbb{P}(1,1,3)}(3d)$ agree for all $d \in \mathbb{Z}$. For other bundles, the correspondence is more subtle. For instance, the Fourier–Mukai transform of the structure sheaf of the exceptional curve on $\mathcal{H}_3$ is supported on the stacky point of $\mathbb{P}(1,1,3)$.

The Bondal–Thomsen collection corresponds to a set of degrees in $\mathrm{Cl}(\mathcal{H}_3) = \mathbb{Z}^2$. Each of these degrees has the form $-d_i$ where d_i is some effective degree. The secondary fan lives in $\mathrm{Cl}(\mathcal{H}_3) \otimes_{\mathbb{Z}} \mathbf{R}$ and has support equal to the effective cone of $\mathcal{H}_3$. So we can overlay the secondary fan with the d_i from the Bondal–Thomsen collection. This is illustrated in Figure 12.2.

To state Theorem 12.5 precisely, we need to assign to each element $-d \in \Theta$ an appropriate object in D_{Cox}. The issue is that we have too many possible choices! For a given $-d$ should we take the pullback of $\mathcal{O}_{\mathbb{P}(1,1,3)}(-d)$ or the pullback of $\mathcal{O}_{\mathcal{H}_3}(-d)$. If these always coincided, we would not have an issue, but we are not that lucky.

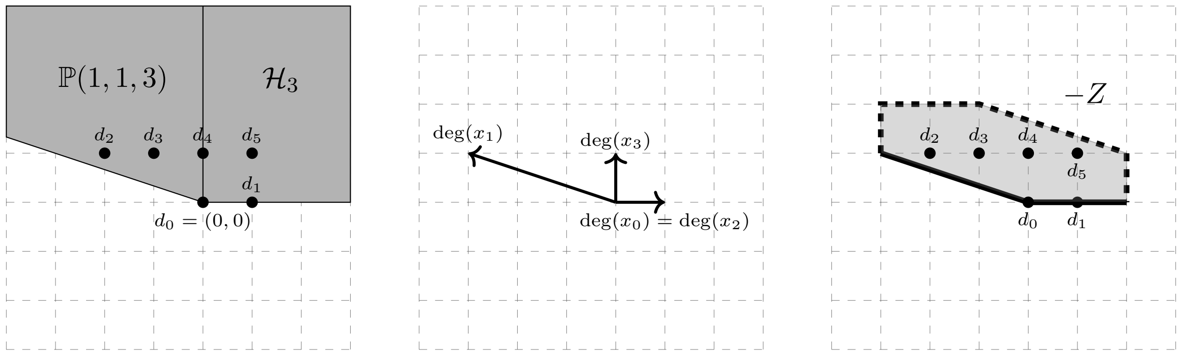


FIGURE 12.2. The secondary fan for a Hirzebruch surface $\mathcal{H}_3$ has two maximal chambers: one corresponding to $\mathcal{H}_3$ and the other to $\mathbb{P}(1,1,3)$. The Bondal–Thomsen collection Θ consists of the degrees in a half-open zonotope Z determined by the degrees of the variables. In this example, Θ consists of the 6 degrees $-d_i$, with d_i as marked in $-Z$.

So for a given degree d, how do we decide when to pull back from $\mathbb{P}(1,1,3)$ and when to pullback from $\mathcal{H}_3$? We let the secondary fan be our guide. For instance, referring to Figure 12.2, d_5 lies in the chamber for $\mathcal{H}_3$; thus, we define $\mathcal{O}_{\mathrm{Cox}}(-d_5)$ as the pullback of $\mathcal{O}_{\mathcal{H}_3}(-d_5)$.

We thus pull back $-d_1$ and $-d_5$ from $\mathcal{H}_3$, and $-d_2$ and $-d_3$ from $\mathbb{P}(1,1,3)$; for $-d_0$ and $-d_4$, either choice will work because the pullbacks coincide.

This example indicates the general recipe.

Definition 12.15. Choose $-d \in \Theta$. Since d is an effective degree, its image in $\mathrm{Cl}(X)_{\mathbf{R}}$ lies in one of the maximal chambers Γ_i of the secondary fan, and this chamber corresponds to some $\mathcal{X}_i$. For this degree, we define $\mathcal{O}_{\mathrm{Cox}}(-d) := \pi_i^* \mathcal{O}_{\mathcal{X}_i}(-d)$.

We also need to choose an ordering on the elements of Θ.

Definition 12.16. The set Θ is equipped with a partial ordering such that $-d \leq -d'$ if and only if $d - d'$ is effective. We choose any refinement of this partial ordering to a total ordering $-d_0, \dots, -d_t$.

Example 12.17. Continuing with the notation of Figure 12.2, the ordering would be $-d_5 < -d_4 < -d_3 < -d_2 < -d_1 < -d_0$

13. Lecture 13 (John Cobb): The Cox category, II: Fourier-Mukai Transforms

Each of the toric stacks $\mathcal{X}_1, \dots, \mathcal{X}_r$ contains a copy of the same dense torus. In particular, there is a natural birational map $\mathcal{X}_i \dashrightarrow \mathcal{X}_j$ for any i, j. We write Γ_{ij} for the graph of this birational map as a substack of $\mathcal{X}_i \times \mathcal{X}_j$. There are natural projection maps p, q from $\mathcal{X}_i \times \mathcal{X}_j$ to the two factors. We obtain the following diagram:

$$\begin{tikzcd} & \mathcal{X}_i \times \mathcal{X}_j \arrow[dl, "p"'] \arrow[dr, "q"] & \\ \mathcal{X}_i & & \mathcal{X}_j. \end{tikzcd}$$

This allows us to define a Fourier–Mukai transform

$$\Phi_{ij} \colon D^b(\mathcal{X}_i) \to D^b(\mathcal{X}_j) \text{ where } \Phi_{ij}(\mathcal{E}) := Rq_* \left(p^* \mathcal{E} \overset{L}{\otimes}_{\mathcal{O}_{\mathcal{X}_i \times \mathcal{X}_j}} \mathcal{O}_{\Gamma_{ij}} \right).$$

At a more concrete level, the Fourier–Mukai transform can be understood as follows. There is a Hanlon-Hicks-Lazarev complex $\mathcal{R}$ that resolves $\Gamma_{ij} \subseteq \mathcal{X}_i \times \mathcal{X}_j$ and so we can "underive" the tensor product, by replacing $\mathcal{O}_{\Gamma_{ij}}$ by $\mathcal{R}$:

$$\Phi_{ij}(\mathcal{E}) = Rq_* \left(p^*\mathcal{E} \otimes_{\mathcal{O}_{\mathcal{X}_i \times \mathcal{X}_j}} \mathcal{R} \right).$$

This provides a geometric way to "move" a sheaf (or an object of the derived category) from one of the $\mathcal{X}_i$ to all of the others. Namely, imagine we have a sheaf $\mathcal{E}_1$ on $\mathcal{X}_1$. For each j, we obtain $\Phi_{1j}(\mathcal{E}_1)$ on $\mathcal{X}_j$. Since the $\mathcal{X}_i$ all agree on the dense torus, $\Phi_{1j}(\mathcal{E}_1)$ will agree with $\mathcal{E}_1$ on the dense torus and will only exhibit different behavior on the toric boundary.

We will not use the following result in this lecture, but it will be used later.

Proposition 13.1. *Let $\widetilde{\mathcal{X}}$ be as in the definition of the Cox category, and let $\pi_i \colon \widetilde{\mathcal{X}} \to \mathcal{X}_i$ be the accompanying maps for all i. There is an isomorphism $\Phi_{ij} \cong R\pi_{j*}\pi_i^*$.*

Sketch of proof. Let $t_i \colon \Gamma_{ij} \to \mathcal{X}_i$ and $t_j \colon \Gamma_{ij} \to \mathcal{X}_j$ be the canonical maps. The first observation is that Φ_{ij} is given by $Rt_{j*}t_i^*$; this follows from the projection formula, in a manner similar to the proof of Proposition 1.8. Now, the claim is that there exists a map $\pi \colon \widetilde{X} \to \Gamma_{ij}$ such that π^* is fully faithful, $t_i\pi = \pi_i$, and $t_j\pi = \pi_j$. Indeed, the fully faithfulness of π^* is equivalent to the unit of adjunction $\mathrm{id} \to R\pi_*\pi^*$ being an isomorphism, and so we have $\Phi_{ij} = Rt_{j*}t_i^* = R(t_j \circ \pi)_*(t_i\pi)^* = R\pi_{j*}\pi_i^*$. The map π may be induced by a refinement of stacky fans, and its fully faithfulness can be deduced from [BBB+24, Lemma 3.7]; we omit the details. ☐

Example 13.2. For any i, j we have $\Phi_{ij}(\mathcal{O}_{\mathcal{X}_i}) = \mathcal{O}_{\mathcal{X}_j}$. Why? The pullback $p^*\mathcal{O}_{\mathcal{X}_i}$ is simply $\mathcal{O}_{\mathcal{X}_i \times \mathcal{X}_j}$; this is a line bundle, and so the derived tensor product with $\mathcal{O}_{\Gamma_{ij}}$ is simply the (underived) tensor product, which is

$$\mathcal{O}_{\mathcal{X}_i \times \mathcal{X}_j} \otimes_{\mathcal{O}_{\mathcal{X}_i \times \mathcal{X}_j}} \mathcal{O}_{\Gamma_{ij}} = \mathcal{O}_{\Gamma_{ij}}.$$

We thus have that $\Phi_{ij}(\mathcal{O}_{\mathcal{X}_i})$ is the pushforward of $\mathcal{O}_{\Gamma_{ij}}$ under the natural birational morphism $\Gamma_{ij} \to \mathcal{X}_j$. We then have $R(q|_{\Gamma_{ij}})_*\mathcal{O}_{\Gamma_{ij}} = \mathcal{O}_{\mathcal{X}_j}$ by [BBB+24, Proposition 3.5].

Example 13.3. If $i = j$ then Γ_{ij} is simply the diagonal $\Delta \subseteq \mathcal{X}_i \times \mathcal{X}_i$. In this case, Φ_{ii} is a Fourier–Mukai transform with respect to a resolution of the diagonal. Thus Φ_{ii} is equivalent to the identity functor (Proposition 1.8).

Example 13.4. Let us return to our running example of the Hirzebruch surface of type 3. Consider

$$\Phi_{21} \colon D(\mathcal{H}_3) \to D(\mathbb{P}(1,1,3)).$$

Let us apply this transform to the Bondal–Thomsen element $\mathcal{O}_{\mathcal{H}_3}(-1,0)$. To perform the computation, we start with the Hanlon-Hicks-Lazarev resolution for Γ_{ij}. This has the form

$$\begin{matrix} \mathcal{O}(0,0,0) \\ \oplus \\ \mathcal{O}(2,-1,-2) \\ \oplus \\ \mathcal{O}(1,-1,-1) \end{matrix} \longleftarrow \begin{matrix} \mathcal{O}(1,-1,-2)^2 \\ \oplus \\ \mathcal{O}(2,-1,-3)^2 \\ \oplus \\ \mathcal{O}(0,-1,-1)^2 \\ \oplus \\ \mathcal{O}(-1,0,-1) \end{matrix} \longleftarrow \begin{matrix} \mathcal{O}(1,-1,-3) \\ \oplus \\ \mathcal{O}(2,-1,-4) \\ \oplus \\ \mathcal{O}(0,-1,-2) \\ \oplus \\ \mathcal{O}(-1,-1,-1) \end{matrix}.$$

Each line bundle is a box product from the factors. For instance, $\mathcal{O}(2,-1,-2)$ means $\mathcal{O}_{\mathcal{H}_3}(2,-1)\boxtimes\mathcal{O}_{\mathbb{P}(1,1,3)}(-2)$. When we pull back $\mathcal{O}_{\mathcal{H}_3}(-1,0)$ and tensor with this complex, we get:

$$\begin{matrix} \mathcal{O}(-1,0,0) \\ \oplus \\ \mathcal{O}(1,-1,-2) \\ \oplus \\ \mathcal{O}(0,-1,-1) \end{matrix} \longleftarrow \begin{matrix} \mathcal{O}(0,-1,-2)^2 \\ \oplus \\ \mathcal{O}(1,-1,-3)^2 \\ \oplus \\ \mathcal{O}(-1,-1,-1)^2 \\ \oplus \\ \mathcal{O}(-2,0,-1) \end{matrix} \longleftarrow \begin{matrix} \mathcal{O}(0,-1,-3) \\ \oplus \\ \mathcal{O}(1,-1,-4) \\ \oplus \\ \mathcal{O}(-1,-1,-2) \\ \oplus \\ \mathcal{O}(-2,-1,-1) \end{matrix}.$$

Call this complex $\mathcal{C}$. There is a hypercohomology spectral sequence

$$E_1^{s,t} = R^t q_*(\mathcal{C}^s) \Rightarrow R^{s+t} q_*(\mathcal{C}). \tag{13.5}$$

As in Construction 10.5, given a coherent sheaf $\mathcal{F}$ on $\mathcal{H}_3$ and a locally free sheaf $\mathcal{G}$ on $\mathbb{P}(1,1,3)$, the projection formula and base change imply that $R^t q_*(\mathcal{F}\boxtimes\mathcal{G}) \cong H^t(\mathcal{H}_3,\mathcal{F})\otimes_{\mathbf{k}}\mathcal{G}$. For all but one summand $\mathcal{L}\boxtimes\mathcal{L}'$ of $\mathcal{C}$, we have $H^t(\mathcal{H}_3,\mathcal{L}) = 0$ for all t; the only nonvanishing cohomology that arises is $H^1(\mathcal{H}_3,\mathcal{O}(-2,0)) \cong \mathbf{k}$. We conclude that $E_1^{s,t} = 0$ unless $s=-1$ and $t=1$, and so $Rq_*(\mathcal{C}) \cong \mathcal{O}_{\mathbb{P}(1,1,3)}(-1)$ (concentrated in cohomological degree 0).

Remark 13.6. There is also an algebraic method to "move" a sheaf from one of the $\mathcal{X}_i$ to all of the others. For instance, as in Example 13.4, let us start with $\mathcal{O}_{\mathcal{H}_3}(-1,0)$ on $\mathcal{H}_3$. We can represent this by a graded S-module: for instance $S(-1,0)$ is a natural choice, as $S(-1,0)|_{\mathcal{H}_3} = \mathcal{O}_{\mathcal{H}_3}(-1,0)$. We can then use $S(-1,0)$ to determine a sheaf on $\mathbb{P}(1,1,3)$ namely: $S(-1,0)|_{\mathbb{P}(1,1,3)} = \mathcal{O}_{\mathbb{P}(1,1,3)}(-1)$. But one should be cautious, as the outcome of this method depends on the module one uses to represent the initial sheaf. For instance, we could also have represented $\mathcal{O}_{\mathcal{H}_3}(-1,0)$ via a different module. A rather artificial choice would be to write $M = S(-1,0)\oplus S/(x_0,x_2)$. Since (x_0,x_2) is a component of the irrelevant ideal for $\mathcal{H}_3$, we have $(S/(x_0,x_2))|_{\mathcal{H}_3} = 0$ as a sheaf on $\mathcal{H}_3$. Thus, $M|_{\mathcal{H}_3} \cong \mathcal{O}_{\mathcal{H}_3}(-1,0)$. But on $\mathbb{P}(1,1,3)$, (x_0,x_2) is not a component of the irrelevant ideal; it is the ideal of the singular point $p = [0:0:1]$. Thus, we have $M|_{\mathbb{P}(1,1,3)} = \mathcal{O}_{\mathbb{P}(1,1,3)}(-1)\oplus\mathcal{O}_p$.

Remark 13.7. What does the Revised King's Conjecture really tell us? Let's first address what it specifically tells us about sheaves and multigraded rings.

We get a uniqueness property akin to Corollary 2.6. Namely, take a sheaf $\mathcal{E}$ on a toric variety X. Because Θ generates $D(X)$, one can represent $\mathcal{E}$ in terms of the Bondal–Thomsen collection, but this representation will often be fundamentally non-unique. However, if we want to represent $\mathcal{E}$ *and* the Fourier–Mukai transforms of $\mathcal{E}$ to the other $\mathcal{X}_i$ from the secondary fan, then we can do this in a unique (up to homotopy) way. This is (very roughly) the statement of Theorem 17.3. In short: in some sense, it is simpler to classify sheaves on the whole secondary fan than it is to just classify sheaves on a single toric variety X.

So the theorem tells us that while King's Conjecture was false, the underlying idea was very much correct. The cases where full strong exceptional collections of line bundles exist—projective space, products, projective bundles, etc—are indicative of what happens in the general case. The correct statement just required a framework that was not yet obvious back then.

14. Lecture 14 (Milena Hering): Proof of revised King's conjecture, I: Outline

The Revised King's Conjecture claims that the line bundles $\mathcal{O}_{\mathrm{Cox}}(-d)$ for $-d$ in the Bondal-Thomsen collection form a full strong exceptional collection. Why would this be true, and how could we prove it?

Let's start with the special case where the secondary fan only has a single chamber, and where the toric variety in that chamber is smooth. We write X for this toric variety. Then $D_{\mathrm{Cox}}(S) = D(X)$ and each $\mathcal{O}_{\mathrm{Cox}}(-d)$ is simply $\mathcal{O}_X(-d)$. Choose two elements $-d, -d'$ in the Bondal-Thomsen collection Θ. We must show three things:

(1) Exceptionality: For any $-d \in \Theta$, we have

$$\mathrm{Hom}_X(\mathcal{O}_X(-d), \mathcal{O}_X(-d)) = \mathbf{k}, \text{ and } \mathrm{Ext}^i_X(\mathcal{O}_X(-d), \mathcal{O}_X(-d)) = 0 \text{ for } i > 0.$$

(2) Orthogonality: For any $-d' < -d \in \Theta$ we have
   (a) $\mathrm{Ext}^i_X(\mathcal{O}_X(-d), \mathcal{O}_X(-d')) = 0$ for $i > 0$ and
   (b) $\mathrm{Hom}(\mathcal{O}_X(-d), \mathcal{O}_X(-d')) = 0$.

(3) Fullness: The bundles $\mathcal{O}_X(-d)$ with $-d \in \Theta$ generate the derived category.

Exceptionality is trivial:

$$\mathrm{Hom}_X(\mathcal{O}_X(-d), \mathcal{O}_X(-d)) \cong H^0(X, \mathcal{O}_X(-d-(-d))) = H^0(X, \mathcal{O}_X) = \mathbf{k}$$

since X is a projective variety. Also,

$$\mathrm{Ext}^i_X(\mathcal{O}_X(-d), \mathcal{O}_X(-d)) \cong H^i(X, \mathcal{O}_X(-d-(-d))) = H^i(X, \mathcal{O}_X) = 0$$

by Demazure Vanishing (Theorem 6.8). Fullness follows from Hanlon–Hicks–Lazarev's proof of Bondal's Conjecture (Conjecture 10.8), which we saw back in §10.2. Let us then turn to orthogonality. Part (b) of orthogonality is built into our definition of the ordering of the elements of Θ; see Definition 12.16. So orthogonality, part (a), is the meat of the argument.

Lemma 14.1. *Suppose X is a smooth projective toric variety whose secondary fan has a single chamber, and say $-d < -d' \in \Theta$. We have*

$$\mathrm{Ext}^i_X(\mathcal{O}_X(-d), \mathcal{O}_X(-d')) = 0 \textit{ for all } i > 0.$$

Proof. We have $\mathrm{Ext}^i_X(\mathcal{O}_X(-d), \mathcal{O}_X(-d)) \cong H^i(X, \mathcal{O}_X(d-d')) = 0$, and $d-d'$ is effective. Since the secondary fan of X has a single chamber, this means $d-d'$ is nef, and so we are done by Demazure vanishing (Theorem 6.8). □

The above outline extends to the case where the secondary fan has one chamber, but where X is simplicial but not smooth, by replacing X with its associated toric stack.

The case above is very special, however, as secondary fans generally have more than one chamber. Nevertheless, it provides something of a roadmap for the general proof. Exceptionality remains trivial. Fullness essentially also follows from Hanlon–Hicks–Lazarev's proof of Bondal's Conjecture, but this is a bit more of a delicate argument. Orthogonality (b) once again follows from the definition of the ordering on Θ. And so, as above, Orthogonality (a) will be the hardest part.

Let's examine what is needed to prove orthogonality. Each $\mathcal{O}_{\mathrm{Cox}}(-d)$ is defined as a pullback from one particular $\mathcal{X}_i$. For orthogonality, we need to understand the relationship between two such bundles. So let's say that $\mathcal{O}_{\mathrm{Cox}}(-d)$ is the pullback of $\mathcal{O}_{\mathcal{X}_1}(-d)$ and $\mathcal{O}_{\mathrm{Cox}}(-d')$ is the pullback of $\mathcal{O}_{\mathcal{X}_2}(-d')$. We need to compute the Hom and Ext between these sheaves, and we thus need a way to compare sheaves pulled back from $\mathcal{X}_1$ with sheaves pulled back from $\mathcal{X}_2$. That is, we need to compute:

$$\mathrm{Ext}^i_{D_{\mathrm{Cox}}(S)}(\mathcal{O}_{\mathrm{Cox}}(-d), \mathcal{O}_{\mathrm{Cox}}(-d')) = \mathrm{Ext}^i_{D_{\mathrm{Cox}}(S)}(\pi_1^*\mathcal{O}_{\mathcal{X}_1}(-d), \pi_2^*\mathcal{O}_{\mathcal{X}_2}(-d'))$$

Adjunction allows us to swap the π_1^* in the first factor to a π_{1*} in the second factor. This yields

$$= \operatorname{Ext}^i_{\mathcal{X}_1}(\mathcal{O}_{\mathcal{X}_1}(-d), \pi_{1*}\pi_2^*\mathcal{O}_{\mathcal{X}_2}(-d')).$$

We have reduced to a computation on $\mathcal{X}_1$, but we need to understand this weird-looking object $\pi_{1*}\pi_2^*\mathcal{O}_{\mathcal{X}_2}(-d')$. But wait, we have seen this before! It is the result of applying the Fourier–Mukai transform Φ_{21} to $\mathcal{O}_{\mathcal{X}_2}(-d')$ (Proposition 13.1). Thus:

$$= \operatorname{Ext}^i_{\mathcal{X}_1}(\mathcal{O}_{\mathcal{X}_1}(-d), \Phi_{21}\mathcal{O}_{\mathcal{X}_2}(-d')).$$

And so the proof of Revised King's Conjecture will require us to understand the effect of Fourier–Mukai transforms on Bondal–Thomsen line bundles. In general, understanding the Fourier–Mukai transform of a sheaf can be very difficult. But it turns out that, for the Bondal–Thomsen elements, the results of the Fourier–Mukai transforms are as simple as one could ever hope for.

Lemma 14.2 (Θ-Transform Lemma)**.** *Let $-d \in \Theta_X$ be an element whose image in Σ_{GKZ} lies in the chamber corresponding to $\mathcal{X}_i$. For any j,*

$$\Phi_{ij}(\mathcal{O}_{\mathcal{X}_i}(-d)) = \mathcal{O}_{\mathcal{X}_j}(-d).$$

Equivalently:

$$\pi_{j*}\mathcal{O}_{\mathrm{Cox}}(-d) = \mathcal{O}_{\mathcal{X}_j}(-d).$$

In particular, the statement says the following: start with a Bondal–Thomsen line bundle $\mathcal{O}_{\mathcal{X}_i}(-d)$ where d comes from the chamber corresponding to $\mathcal{X}_i$. When you Fourier–Mukai transform this bundle to another chamber, you will get the corresponding Bondal–Thomsen line bundle on the nose. That is, $\mathcal{O}_{\mathcal{X}_i}(-d)$ transforms into $\mathcal{O}_{\mathcal{X}_j}(-d)$.

There are two miracles here, neither of which is obvious at first. The first miracle is that the transform of this sheaf is a sheaf, i.e. that the higher pushforwards R^i all vanish. This is not generally true for Fourier–Mukai transforms, and so it is special for this case. The second miracle is that the transform of a line bundle is still a line bundle. Again, this is not generally true for Fourier–Mukai transforms.

In short, Lemma 14.2 is quite subtle, and it fails if one alters the hypotheses even slightly. We will explore this delicacy in more detail in Lecture 15. But we emphasize that the challenge in the lemma has nothing to do with stackiness, as the statement is equally subtle when both $\mathcal{X}_i$ and $\mathcal{X}_j$ are smooth toric varieties.

Remark 14.3. As far as we were able to determine, the Θ-Transform Lemma does not appear to easily follow from known vanishing results from toric or birational geometry. However, it would be interesting to explore this in greater detail.

The next lecture will tackle the proof of the Θ-Transform Lemma. Before we dive into that, we want to note that the lemma almost immediately implies the orthogonality statement we needed. Combining the above computation with the Θ-Transform Lemma, we conclude:

$$\operatorname{Ext}^i_{D_{\mathrm{Cox}}(S)}(\mathcal{O}_{\mathrm{Cox}}(-d), \mathcal{O}_{\mathrm{Cox}}(-d')) = \operatorname{Ext}^i_{\mathcal{X}_1}(\mathcal{O}_{\mathcal{X}_1}(-d), \mathcal{O}_{\mathcal{X}_1}(-d')) = H^i(\mathcal{X}_1, \mathcal{O}_{\mathcal{X}_1}(d-d')).$$

Since $-d$ comes from the chamber Γ_1, it follows that $\mathcal{O}_{\mathcal{X}_1}(d)$ is nef. By Demazure Vanishing for $\mathbb{Q}$-Cartier divisors (Theorem 6.8), we conclude that all of these Ext groups vanish. In more detail: write $-d = \sum_\rho a_\rho D_\rho$ and $-d' = \sum_\rho \lfloor b_\rho \rfloor D_\rho$, where $a_\rho \in \mathbb{Z}$, and $b_\rho \in (-1, 0]$. The divisor $\sum_\rho b_\rho D_\rho$ is $\mathbb{Q}$-linearly equivalent to zero, and so $-d$ is $\mathbb{Q}$-linearly equivalent to $\sum_\rho (a_\rho + b_\rho) D_\rho$. In particular, $\sum_\rho (a_\rho + b_\rho) D_\rho$ is a nef $\mathbb{Q}$-divisor. Since each a_ρ is an integer, rounding down the coefficients of $\sum_\rho (a_\rho + b_\rho) D_\rho$ gives $\sum_\rho (a_\rho + \lfloor b_\rho \rfloor) = -d - d'$. Now apply Theorem 6.8.

In short: combining the Θ-Transform Lemma and Demazure Vanishing immediately yields the orthogonality statement we need.

Example 14.4. Let us revisit the example of $\mathcal{H}_3$, the Hirzebruch surface of type 3. Recall that $(-1,0)$ and $(2,-1)$ are both in the Bondal-Thomsen collection of $\mathcal{H}_3$ (see Figure 12.2). Since $(-1,0)-(2,-1)=(-3,1)$ is effective, we have $(-1,0)>(2,-1)$, and so we should get $\mathrm{RHom}(\mathcal{O}_{\mathrm{Cox}}(-1,0),\mathcal{O}_{\mathrm{Cox}}(2,-1))=0$. We now check this directly. Recall that the two chambers of the secondary fan correspond to $\mathcal{H}_3$ and $\mathbb{P}(1,1,3)$. Since $(1,0)$ lies in the chamber corresponding to $\mathcal{H}_3$, we have $\mathcal{O}_{\mathrm{Cox}}(-1,0)=\pi_1^*\mathcal{O}_{\mathcal{H}_3}(-1,0)$. Since $d=(-2,1)$ lies in the other chamber, we must use the fact that there is a natural corresponding element in $\Theta_{\mathbb{P}(1,1,3)}$. In this case, the map $\alpha\colon \mathbb{Z}^2\to \mathrm{Cl}(\mathbb{P}(1,1,3))$ is given by $(a,b)\mapsto a+3b$ and thus $\mathcal{O}_{\mathrm{Cox}}(2,-1)=\pi_2^*\mathcal{O}_{\mathbb{P}(1,1,3)}(\alpha(2,-1))=\pi_2^*\mathcal{O}_{\mathbb{P}(1,1,3)}(-1)$. We now compute:

$$\begin{aligned}
\mathrm{RHom}(\mathcal{O}_{\mathrm{Cox}}(-1,0),\mathcal{O}_{\mathrm{Cox}}(2,-1)) &= \mathrm{RHom}(\pi_1^*\mathcal{O}_{\mathcal{H}_3}(-1,0),\pi_2^*\mathcal{O}_{\mathbb{P}}(2,-1)) && \text{by definition,}\\
&= \mathrm{RHom}(\mathcal{O}_{\mathcal{H}_3}(-1,0),\pi_{1*}\pi_2^*\mathcal{O}_{\mathbb{P}}(2,-1)) && \text{by adjunction,}\\
&= \mathrm{RHom}(\mathcal{O}_{\mathcal{H}_3}(-1,0),\mathcal{O}_{\mathcal{H}_3}(2,-1)) && \text{by Lemma 14.2,}\\
&= \mathrm{RHom}(\mathcal{O}_{\mathcal{H}_3},\mathcal{O}_{\mathcal{H}_3}(3,-1))=0 && \text{by direct computation.}
\end{aligned}$$

Note that RHom in the opposite order is nonzero:

$$\begin{aligned}
\mathrm{RHom}(\mathcal{O}_{\mathrm{Cox}}(2,-1),\mathcal{O}_{\mathrm{Cox}}(-1,0)) &= \mathrm{RHom}(\pi_2^*\mathcal{O}_{\mathbb{P}(1,1,3)}(-1),\pi_1^*\mathcal{O}_{\mathcal{H}_3}(-1,0)) && \text{by definition,}\\
&= \mathrm{RHom}(\mathcal{O}_{\mathbb{P}(1,1,3)}(-1),\pi_{2*}\pi_1^*\mathcal{O}_{\mathcal{H}_3}(-1,0)) && \text{by adjunction,}\\
&= \mathrm{RHom}(\mathcal{O}_{\mathbb{P}(1,1,3)}(-1),\mathcal{O}_{\mathbb{P}(1,1,3)}(-1)) && \text{by Lemma 14.2,}\\
&= \Bbbk && \text{by direct computation.}
\end{aligned}$$

Through similar computations for the other pairs, one can directly confirm that the objects $\mathcal{O}_{\mathrm{Cox}}(-d)$ for $-d\in\Theta$ form a strong exceptional collection of $D_{\mathrm{Cox}}(S)$.

To summarize, the outline of the proof amounts to:

(1) Combine the Θ-Transform Lemma and Demazure Vanishing to obtain the necessary orthogonality statement.
(2) Use the Hanlon-Hicks-Lazarev resolution of the diagonal to conclude that $\mathcal{O}_{\mathrm{Cox}}(-d)$ with $-d\in\Theta$ generate $D_{\mathrm{Cox}}(S)$.

15. Lecture 15 (Patrick Kennedy-Hunt): Proof of revised King's conjecture, II: Exceptionality

As discussed in the previous section, when it comes to proving the orthogonality properties of the Bondal–Thomsen collection, the Θ-Transform Lemma is the key. Let us recall the setup. Choose some degree $-d\in\Theta$ where d lies in the secondary fan chamber corresponding to $\mathcal{X}_1$. We have the following diagram:

$$\begin{tikzcd} & \Gamma_{12} \arrow[dl, "p"'] \arrow[dr, "q"] & \\ \mathcal{X}_1 & & \mathcal{X}_2. \end{tikzcd}$$

And we must prove that

$$R^\ell q_*(p^*\mathcal{O}_{\mathcal{X}_1}(-d))=\begin{cases}\mathcal{O}_{\mathcal{X}_2}(-d) & \text{if } \ell=0\\ 0 & \text{if } \ell>0.\end{cases}$$

This breaks into fundamentally different parts:

- **R^0-term:** we must show that the R^0q_*-term is the correct line bundle.
- **Vanishing:** we must prove the vanishing of the higher direct images $R^\ell q_*$ with $\ell>0$.

As we will try to demonstrate in the next example, these two parts are in tension with one another.

Example 15.1 (Atiyah Flop)**.** This is a non-projective example, but it is instructive for thinking about the Θ-Transform Lemma. Consider the Cox ring $S = \Bbbk[x_0, x_1, y_0, y_1]$ with $\deg(x_i) = 1$ and $\deg(y_j) = -1$. The Bondal zonotope (see Definition 6.4) associated to this nonstandard graded ring is the open interval $(-2, 2)$, and so the Bondal–Thomsen collection consists of the divisors of degree $-1, 0$ and 1.

The secondary fan has two chambers: $\mathbb{R}_{\geq 0}$ and $\mathbb{R}_{\leq 0}$. The first chamber corresponds to a threefold $Y_+ = [\mathrm{Spec}(S) - V(x_0, x_1)/\mathbb{G}_m]$. This is a $\mathbb{P}^1$-bundle (coordinates x_0, x_1 on the fibers) over $\mathbb{A}^2$ (coordinates y_0, y_1 on the base). The second chamber corresponds to $Y_- = [\mathrm{Spec}(S) - V(y_0, y_1)/\mathbb{G}_m]$, which is a $\mathbb{P}^1$-bundle (coordinates y_0, y_1 on the fibers) over $\mathbb{A}^2$ (coordinates x_0, x_1 on the base). Let $\widetilde{Y}$ be the blowup of Y_+ at the line $V(y_0, y_1)$; this is isomorphic to the blowup of Y_- at $V(x_0, x_1)$. So we have a diagram:

$$\begin{tikzcd} & \widetilde{Y} \arrow[dl, "p"'] \arrow[dr, "q"] & \\ Y_+ & & Y_-. \end{tikzcd}$$

As we often do, we will express sheaves on Y_+ and Y_- via the corresponding S-modules. We have $\mathrm{Pic}(Y_+) = \mathbb{Z}$, and for $d \in \mathbb{Z}$, we write $\mathcal{O}_{Y_+}(d)$ for the line bundle associated to the free module $S(d)$. Let us consider the Fourier–Mukai transforms $R_{q*}p^*\mathcal{O}_{Y_+}(d)$ for all d.

We start by computing the R^0-term. For any d we have

$$R^0 q_* p^* \mathcal{O}_{Y_+}(d) = \widetilde{S_{\geq d}}(d).$$

Note that this is a line bundle if and only if $d \leq 0$. More specifically, for $d \leq 0$ we have $S_{\geq d} = S$ and thus $R^0 q_* \mathcal{O}_{Y_+}(d) = \widetilde{S}(d) = \mathcal{O}_{Y_-}(d)$. But for $d > 0$, we have $S_{\geq d} = \langle x_0, x_1\rangle^d$. Since $\langle x_0, x_1\rangle$ is *not* a component of the irrelevant ideal on Y_-, this induces a nontrivial ideal sheaf on Y_-. In particular, we have $R^0 q_* \mathcal{O}_{Y_+}(d) = \mathcal{O}_{Y_-}(d)$ if and only if $d \leq 0$. Thus, obtaining the correct R^0-term requires the initial line bundle to be "*not too positive*".

Let us now turn to the higher direct images. The map $\widetilde{Y} \to Y_-$ is birational over $Y_- \setminus V(x_0, x_1)$, and the fibers over $V(x_0, x_1)$ are 1-dimensional. Thus the $R^\ell q_*$ terms will automatically vanish for all $\ell > 1$, so we only need to be concerned about R^1. A straightforward, if somewhat involved, computation shows that

$$R^1 q_* p^* \mathcal{O}_{Y_+}(d) \neq 0 \iff H^1(Y_+, \mathcal{O}_{Y_+}(d)) \neq 0 \iff H^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(d)) \neq 0 \iff d \leq -2.$$

So we have the vanishing of the higher direct images if and only if the initial line bundle is "*not too negative*".

Let us summarize. Both conclusions—the correct R^0 and the vanishing of the higher direct images—are only satisfied for $d = 0$ and $d = -1$. These are precisely the elements of the Bondal–Thomsen collection whose negatives lie in the chamber of the secondary fan corresponding to Y_+.

The example demonstrates that we require a delicate balance in order to obtain the conclusion of the Θ-Transform Lemma. A complete proof of the Θ-Transform Lemma is given in [BBB+24, §4.3]. Instead of presenting the details of the proof in these notes, we will try instead to capture what aspects of the lemma are nontrivial.

Let us set some notation. We start with a line bundle $\mathcal{O}_{\mathcal{X}_1}(-d)$ on $\mathcal{X}_1$ and where d lies in the chamber Γ_1 of the secondary fan. This implies that $\mathcal{O}_{\mathcal{X}_1}(d)$ is a nef line bundle on $\mathcal{X}_1$, and hence that $\mathcal{O}_{\mathcal{X}_1}(-d)$ is anti-nef. Write $L := p^*\mathcal{O}_{\mathcal{X}_1}(-d)$ for the corresponding line bundle on the graph Γ_{12} and note that, since nefness is stable under pullback, L is an anti-nef line bundle on Γ_{12}.

Let us start with the R^0-term. We want to show that $q_*L \cong \mathcal{O}_{\mathcal{X}_2}(-d)$. This is done in [BBB+24, Lemma 4.14] by comparing the global sections over each cone σ in the fan Σ_2 corresponding to $\mathcal{X}_2$. We can compute the global sections in terms of lattice points satisfying certain inequalities involving piecewise linear support functions. One key computation, given in [BBB+24, Proposition 4.2], is that when d lies in the chamber Γ_1, then the corresponding support function for $\mathcal{O}_{\mathcal{X}_1}(-d)$ "majorizes" the support function for $\mathcal{O}_{\mathcal{X}_j}(-d)$ for any other j. This shows that each section coming from $\mathcal{O}_{\mathcal{X}_2}(-d)$ also yields a section from q_*L.

The opposite direction is more subtle. It relies on the fact that the Bondal–Thomsen line bundles can be expressed as toric divisors with purely fractional coefficients, and then utilizes this to create an inequality related to support functions of the form $\alpha \leq \alpha' \leq \lceil \alpha \rceil$. The key trick is then: if α is rational and α' is integral, then this inequality implies that $\alpha' = \lceil \alpha \rceil$. Again, we refer to [BBB+24, Lemma 4.14] for the details, but we remark that the computation, while a bit longwinded, only relies on elementary facts about toric geometry and the behavior of the ceiling function.

The vanishing of the higher direct pushforwards is more involved. The higher direct image sheaves $R^\ell q_* L$ encode the higher cohomology groups of L restricted to the fibers of $q\colon \Gamma_{12} \to \mathcal{X}_2$. We would immediately obtain the vanishing of these if L satisfied an appropriate positivity condition. For instance, if L were a nef line bundle then a relative version of Demazure vanishing would imply that the higher direct images vanish. However L is not nef: in fact, as noted above, L is anti-nef.

So to avoid having nonzero higher direct images, it is crucial that L be *not too negative*. Here is where the Bondal–Thomsen condition must come into play. Recall that L is the pullback of the Bondal–Thomsen element $\mathcal{O}_{\mathcal{X}_1}(-d)$, which is a divisor whose coefficients lie in $(-1, 0]$.

If L itself were an element of the Bondal–Thomsen collection of Γ_{12}, then Demazure Vanishing would again imply that the higher direct images vanish. However, it is *not necessarily* the case that the pullback of a Bondal–Thomsen element is a Bondal–Thomsen element. The pullback can acquire coefficients along exceptional divisors that are not in the range $(-1, 0]$.

In fact, the Θ-Transform Lemma holds true even if one replaces Γ_{12} by some larger, birational toric cover. This is the fundamentally most subtle aspect of the proof: we cannot easily control what happens to the coefficients of the toric divisor L on Γ_{12}, and so it is difficult to utilize "off-the-shelf" vanishing statements, like Kawamata–Viehweg Vanishing. The proof in [BBB+24, §4.3] utilizes a polytope method, due to [ABKW20, Theorem 3.6], to compute the higher cohomology of line bundles of the form $D - D'$ where both D and D' are nef.

Remark 15.2. Often, the first proofs of an essential statement prove to be more complicated than necessary. This happened with Hanlon–Hicks–Lazarev's proofs involving their resolutions, as their proof was later simplified by both [BCHSY25] and [BH25]. We expect that something similar will happen with the Θ-Transform Lemma at some point, i.e. the proof outlined above will not be the "final" proof.

16. Lecture 16 (John Nolan): Proof of Revised King Conjecture, III: Conclusion

In this lecture, we complete the proof of the Revised King Conjecture (Theorem 12.5). We recall our outline of what we need to prove.

(1) Exceptionality: For any $-d \in \Theta$ we have
$$\mathrm{Hom}_{D_{\mathrm{Cox}}(S)}(\mathcal{O}_{\mathrm{Cox}}(-d), \mathcal{O}_{\mathrm{Cox}}(-d)) = \mathbf{k}, \text{ and } \mathrm{Ext}^i_{D_{\mathrm{Cox}}}(\mathcal{O}_{\mathrm{Cox}}(-d), \mathcal{O}_{\mathrm{Cox}}(-d)) = 0 \text{ for } i > 0.$$
(2) Orthogonality: For any $-d' < -d \in \Theta$, we have
  (a) $\mathrm{Ext}^i_{D_{\mathrm{Cox}}(S)}(\mathcal{O}_{\mathrm{Cox}(S)}(-d), \mathcal{O}_{\mathrm{Cox}(S)}(-d')) = 0$ for $i > 0$ and
  (b) $\mathrm{Hom}(\mathcal{O}_{\mathrm{Cox}(S)}(-d), \mathcal{O}_{\mathrm{Cox}(S)}(-d')) = 0$.
(3) Fullness: The bundles $\mathcal{O}_{\mathrm{Cox}(S)}(-d)$ with $-d \in \Theta$ generate $D_{\mathrm{Cox}}(S)$.

Proof of Exceptionality: Let $-d \in \Theta$, and say that d lies in the chamber Γ_i. We have:
$$\begin{aligned}\mathrm{Hom}_{D_{\mathrm{Cox}}(S)}(\mathcal{O}_{\mathrm{Cox}}(-d), \mathcal{O}_{\mathrm{Cox}}(-d)) &= \mathrm{Hom}_{D_{\mathrm{Cox}}(S)}(\pi_i^*\mathcal{O}_{\mathcal{X}_i}(-d), \pi_i^*\mathcal{O}_{\mathcal{X}_i}(-d)) && \text{(Definition 12.15)}\\ &= \mathrm{Hom}_{\mathcal{X}_i}(\mathcal{O}_{\mathcal{X}_i}(-d), \pi_{i*}\pi_i^*\mathcal{O}_{\mathcal{X}_i}(-d)) && \text{(Adjunction)}\end{aligned}$$

By [BBB+24, Lemma 3.7], $\pi_{i*}\pi_i^*$ is the identity; equivalently, π_i^* is fully faithful. So we have

$$\begin{aligned} &= \mathrm{Hom}_{\mathcal{X}_i}(\mathcal{O}_{\mathcal{X}_i}(-d), \mathcal{O}_{\mathcal{X}_i}(-d)) && \text{By [BBB}^+\text{24, Lemma 3.7]} \\ &= H^0(\mathcal{X}_i, \mathcal{O}_{\mathcal{X}_i}) = \mathbf{k}. \end{aligned}$$

The last equality is because $\mathcal{X}_i$ is integral and proper.[9] The computation for the higher Ext groups is similar and boils down to the fact that

$$H^p(\mathcal{X}_i, \mathcal{O}_{\mathcal{X}_i}) = H^p(X_i, \mathcal{O}_{X_i}) = 0 \text{ for } p > 0,$$

which follows from Demazure Vanishing.

Orthogonality (a): Choose $-d$ and $-d'$ in Θ and let Γ_i be a chamber whose closure contains d and let Γ_j be a chamber whose closure contains d'. We have

$$\begin{aligned} \mathrm{Ext}^p\, D_{\mathrm{Cox}}(S)(\mathcal{O}_{\mathrm{Cox}}(-d), \mathcal{O}_{\mathrm{Cox}}(-d')) &= \mathrm{Hom}_{D_{\mathrm{Cox}}(S)}(\pi_i^*\mathcal{O}_{\mathcal{X}_i}(-d), \pi_j^*\mathcal{O}_{\mathcal{X}_i}(-d')) && \text{(Definition 12.15)} \\ &= \mathrm{Ext}^p\, \mathcal{X}_i(\mathcal{O}_{\mathcal{X}_i}(-d), \pi_{i*}\pi_j^*\mathcal{O}_{\mathcal{X}_i}(-d')) && \text{(Adjunction)} \\ &= \mathrm{Ext}^p\, \mathcal{X}_i(\mathcal{O}_{\mathcal{X}_i}(-d), \mathcal{O}_{\mathcal{X}_i}(-d')) && \text{(Lemma 14.2)} \\ &= H^p(\mathcal{X}_i, \mathcal{O}_{\mathcal{X}_i}(d-d')) = 0. \end{aligned}$$

The final equality follows from Demazure Vanishing (Theorem 6.8), in a manner similar to the outline of the proof of Theorem 12.5 in Lecture 14: see the paragraph below Remark 14.3.[10]

Orthogonality (b): This follows from Definition 6.2.

Fullness: This is the last remaining piece. We need to show:

Proposition 16.1. *The Cox category $D_{\mathrm{Cox}}(S)$ is generated by $\mathcal{O}_{\mathrm{Cox}}(-d)$ for $-d \in \Theta$.*

Before proving this, we begin with a heuristic for why it ought to be true, and some approaches for proving it. By Hanlon-Hicks-Lazarev's proof of Bondal's Conjecture (Conjecture 10.8), the line bundles corresponding Bondal–Thomsen degrees generate $D(\mathcal{X}_j)$ for all j. And by the Θ-Transform Lemma, the line bundles $\mathcal{O}_{\mathrm{Cox}}(-d)$ with $-d \in \Theta$ push forward to the Bondal–Thomsen line bundles on $\mathcal{X}_j$ for all j. In other words, the line bundles $\mathcal{O}_{\mathrm{Cox}}(-d)$ with $-d \in \Theta$ push forward to a generating set on $D(\mathcal{X}_j)$ for each j. Since $D_{\mathrm{Cox}}(S)$ was built as the "union", in some sense, of the $D(\mathcal{X}_j)$, this suggests Proposition 16.1 should hold.

But to make this intuition precise, we require an additional argument. There are several ways to proceed.

The most satisfying approach is to enrich $D_{\mathrm{Cox}}(S)$ to a dg-category, and then prove that the Hanlon–Hicks–Lazarev resolution of the diagonal induces a dg-category resolution of the diagonal for $D_{\mathrm{Cox}}(S)$. This is true, and it is implied by Theorem 17.6, but that proof involves a fair amount of category theory. That said, this approach is satisfying because it "explains" the mysterious uniformity of the Hanlon–Hicks–Lazarev resolution of the diagonal observed back in Corollary 8.3, where the resolution is uniform for the X_i appearing in the secondary fan. We will explore this further in Lecture 17.

Another satisfying method would be the following. By the proof Bondal's Conjecture, we know that the Bondal–Thomsen collection $\Theta_{\widetilde{\mathcal{X}}}$ for $\widetilde{\mathcal{X}}$ generates $D(\widetilde{\mathcal{X}})$. If $D_{\mathrm{Cox}}(S) \subseteq D(\widetilde{\mathcal{X}})$ is an admissible subcategory, then we can apply the resulting projection functor $p\colon D(\widetilde{\mathcal{X}}) \to D_{\mathrm{Cox}}(S)$, to conclude that $p(\Theta_{\widetilde{\mathcal{X}}})$ generates $D_{\mathrm{Cox}}(S)$. And then, by utilizing a standard fact that "Bondal–Thomsen

[9]Strictly speaking, we are using a stacky version of this classical fact. Namely, $H^0(\mathcal{X}_i, \mathcal{O}_{\mathcal{X}_i}) = H^0(X_i, \mathcal{O}_{X_i}) = \mathbf{k}$ where the first equality is because $\mathcal{X}_i \to X_i$ is a good moduli space, in the sense of [Alp13, Section 1.2]; see [EM12, Proposition 3.5]. The second follows since X_i is integral and proper.

[10]We are using here the stacky version of Demazure Vanishing. But the shift from varieties to stacks is mostly formal in this setting. See [BBB+24, Theorem 2.11] for the necessary stacky statement.

pushes forward to Bondal–Thomsen" as in, e.g. [BBB+24, Proposition 2.28], we get the desired generation result. Again, this is true, but we will not prove it: see [BBB+24, Corollary 5.5].

Let us now give a proof of Proposition 16.1, following [BBB+24, Proposition 5.1].

Proof. We start by choosing some $m \in M \otimes \mathbb{Q}$ and we write $-\widetilde{d}$ for the corresponding Bondal–Thomsen element in $\mathrm{Cl}(\widetilde{\mathcal{X}})$ and $-d$ for the corresponding Bondal–Thomsen element in $\mathrm{Cl}(\widetilde{\mathcal{X}})$.[11] By [BBB+24, Proposition 2.28], $\pi_{j*}(\mathcal{O}_{\widetilde{\mathcal{X}}}(-\widetilde{d})) = \mathcal{O}_{\mathcal{X}_j}(-d)$. If we choose j so that $\mathcal{O}_{\mathrm{Cox}}(-d) = \pi_j^* \mathcal{O}_{\mathcal{X}_j}(-d)$, then by adjunction we have

$$\begin{aligned} \mathrm{Hom}_{\widetilde{\mathcal{X}}}(\mathcal{O}_{\mathrm{Cox}}(-d), \mathcal{O}_{\widetilde{\mathcal{X}}}(-\widetilde{d})) &= \mathrm{Hom}_{\widetilde{\mathcal{X}}}(\pi_j^* \mathcal{O}_{\mathcal{X}_j}(-d), \mathcal{O}_{\widetilde{\mathcal{X}}}(-\widetilde{d})) \\ &= \mathrm{Hom}_{\widetilde{\mathcal{X}}_j}(\mathcal{O}_{\mathcal{X}_j}(-d), \pi_{j*} \mathcal{O}_{\widetilde{\mathcal{X}}}(-\widetilde{d})) \\ &= \mathrm{Hom}_{\widetilde{\mathcal{X}}_j}(\mathcal{O}_{\mathcal{X}_j}(-d), \mathcal{O}_{\mathcal{X}_j}(-d)). \end{aligned}$$

This last group contains the identity morphism, and so there exists a corresponding $f\colon \mathcal{O}_{\mathrm{Cox}}(-d) \to \mathcal{O}_{\widetilde{\mathcal{X}}}(-\widetilde{d})$. Let $C(f)$ be the cone of f. By the Θ-Transform Lemma and [BBB+24, Proposition 2.28], if we push this forward to $\mathcal{X}_i$ for any i, then we will get

$$\pi_{i*}(C(f)) = \left[\mathcal{O}_{\mathcal{X}_i}(-d) \xrightarrow{\pi_{i*}(f)} \mathcal{O}_{\mathcal{X}_i}(-d) \right].$$

Since the $\pi_{i*}(f)$ maps all agree on the torus (where the $\mathcal{X}_i$ and $\widetilde{\mathcal{X}}$ are all isomorphic), these all agree with the identity map on the dense torus. But a map from a line bundle to itself on $\mathcal{X}_i$ is given by a constant function, and so if it is the identity on a dense open subset, then it is the identity morphism. In particular, $\pi_{i*}(C(f)) = 0$ for all i.

Define $\mathcal{Q}$ to be the subcategory of $D(\widetilde{\mathcal{X}})$ given by objects $\mathcal{F}$ such that $\pi_{i*}\mathcal{F} = 0$ for all i. We also define the left-orthogonal subcategory of $\mathcal{Q}$ as

$${}^{\perp}\mathcal{Q} := \{\mathcal{E} \in D(\widetilde{\mathcal{X}}) : \ \mathrm{Hom}(\mathcal{E}, \mathcal{F}) = 0 \text{ for all } \mathcal{F} \in \mathcal{Q}\}.$$

We have shown that the object $C(f)$ lies in $\mathcal{Q}$. We also have that $D_{\mathrm{Cox}}(S) \subseteq {}^{\perp}\mathcal{Q}$ by a simple adjunction computation. It follows that the composition $D_{\mathrm{Cox}}(S) \to D(\widetilde{\mathcal{X}}) \xrightarrow{\rho} D(\widetilde{\mathcal{X}})/\mathcal{Q}$ is fully faithful. Thus, to prove that the $\mathcal{O}_{\mathrm{Cox}}(-d)$ generate $D_{\mathrm{Cox}}(S)$, it suffices to prove that they generate $D(\widetilde{\mathcal{X}})/\mathcal{Q}$. Since $C(f) \in \mathcal{Q}$, it follows that $\rho(\mathcal{O}_{\widetilde{\mathcal{X}}}(-\widetilde{d})) = \rho(\mathcal{O}_{\mathrm{Cox}}(-d))$. And since the $\mathcal{O}_{\widetilde{\mathcal{X}}}(-\widetilde{d})$ generate $D(\widetilde{\mathcal{X}})$, their images–which equal the Bondal–Thomsen collection for the Cox category–generate $D(\widetilde{\mathcal{X}})/\mathcal{Q}$. ∎

[11] So using notation from §6, we would have that $-d = \deg\lfloor \alpha(m) \rfloor$.

17. Lecture 17 (Mykola Sapranov): Some consequences of the revised King conjecture

We next explore several consequences of the main theorem of [BBB+24] and its proof.

17.1. **Smoothness, properness, and Rouquier dimension of the Cox category.** A $\mathbf{k}$-linear triangulated category $\mathcal{T}$ is called proper if $\sum_{i \in \mathbb{Z}} \dim_{\mathbf{k}} \operatorname{Hom}_{\mathcal{T}}(T, T'[i]) < \infty$ for all $T, T' \in \mathcal{T}$. There is also a categorical notion of smoothness, but it is probably not helpful to give the definition in a talk: we say $\mathcal{T}$ is smooth if $\mathcal{T}$ possesses a dg-enhancement whose diagonal bimodule is perfect. Let us not dwell on this latter definition. There are two points to take away here: (1) the category of perfect complexes on a separated scheme X of finite type over $\mathbf{k}$ is smooth (resp. proper) if and only if X is smooth (resp. proper) (see the proof of [Orl16, Proposition 3.31]) and (2) smooth and proper triangulated categories enjoy many of the same properties as derived categories of smooth and proper varieties. One illustration of (2) is the extensive literature on Hodge theory of smooth and proper dg-categories, as originally proposed by Katzarkov-Kontsevich-Pantev [KKP08].

Theorem 17.1. *Let X be a smooth projective toric variety. The category $D_{\mathrm{Cox}}(X)$ is smooth and proper, and its Rouquier dimension (see Lecture 10 for the definition) is* $\dim(X)$.

Proof. Choose a smooth toric stack $\mathcal{X}$ as in the definition of $D_{\mathrm{Cox}}(X)$, so that $D_{\mathrm{Cox}}(X)$ is an admissible triangulated subcategory of $D(\mathcal{X})$. The stack $\mathcal{X}$ is projective, in the sense of [BLS16, Definition 2.5], and it is tame in the sense of [BLS16, Definition 2.1]. It follows from [BLS16, Theorem 6.4] that $D(\mathcal{X})$ is smooth and proper. The properness of $D_{\mathrm{Cox}}(X)$ is an immediate consequence, and smoothness follows from [Orl16, Proposition 3.8, Theorem 3.25]. As for Rouquier dimension, recall that there are fully faithful embeddings of admissible subcategories $D^b(X) \hookrightarrow D_{\mathrm{Cox}}(X) \hookrightarrow D^b(\mathcal{X})$, and so there are inequalities $\dim D^b(X) \leq \dim D_{\mathrm{Cox}}(X) \leq \dim D^b(\mathcal{X})$ [EL21, Section 3.2]. Finally, we have

$$\dim D^b(X) = \dim X = \dim \mathcal{X} = \dim D^b(\mathcal{X}),$$

where the first and third equalities follow from [FH23, HHL24]. ☐

Remark 17.2. Loosely speaking, it is a general mantra that smooth and proper triangulated categories should be "geometric" in some sense. Of course, $D_{\mathrm{Cox}}(X)$ is of geometric origin: it is by definition a subcategory of the derived category of a toric stack. It is an open question as to whether $D_{\mathrm{Cox}}(X)$, or its invariants, can be given an intrinsic algebro-geometric description.

The intuition is that Cox category is the derived category of a "mythical" toric variety whose Cox ring is S and whose effective cone and nef cone coincide. Since $D_{\mathrm{Cox}}(X)$ is not closed under tensor products, it is not (monoidally) equivalent to the derived category of a smooth projective scheme or stack, and so it is unlikely that this intuition can be made fully rigorous. But nevertheless, we can ask: do its K-theory, Hochschild homology, etc. admit geometric interpretations? For example, singularity categories of hypersurfaces do not have a monoidal structure, but their K-theory and Hochschild homology nevertheless recover classical geometric invariants of the hypersurface singularity: this has been an active area of research over the past two decades [BD20, BFK14a, BFK14b, BP25, BRTV18, BW20a, BW20b, BW22, CT13, Dyc11, Efi18, HLP20, KP21, KK24, Pip22, PV12, Seg13, Shk14, Shk16].

17.2. **Homotopy category description of the Cox category.** Let X be a smooth projective toric variety with Cox ring S, and let $K_\Theta(S)$ denote the homotopy category of bounded complexes of $\mathrm{Cl}(X)$-graded free S-modules whose terms are direct sums of modules of the form $S(d)$, where d lies in the Bondal-Thomsen collection Θ. By [BBB+24, Corollary 4.28] (which is used in the proof that $D_{\mathrm{Cox}}(X)$ has a full strong exceptional collection), there is a natural isomorphism

$$S_{d'-d} \xrightarrow{\cong} \operatorname{Hom}_{\mathrm{Cox}}(\mathcal{O}_{\mathrm{Cox}}(d), \mathcal{O}_{\mathrm{Cox}}(d'))$$

for all $d, d' \in \Theta$. It follows that there is a fully faithful functor $\Psi\colon K_\Theta(S) \to D_{\mathrm{Cox}}(X)$ that sends $S(d)$ to $\mathcal{O}_{\mathrm{Cox}}(d)$ for all $d \in \Theta$.

Theorem 17.3. *The functor Ψ is an equivalence.*

Proof. Since $D_{\mathrm{Cox}}(X)$ is generated by $\mathcal{O}_{\mathrm{Cox}}(d)$ for $d \in \Theta$, this is immediate. □

Resolution of the diagonal for the Cox category. Let X be a smooth projective toric variety with Cox ring S. As emphasized above, a major source of motivation for the revised King's Conjecture is that the HHL resolution of the diagonal of X depends only on the rays of the fan determining X. That is, the resolution of the diagonal looks the same for every toric stack $\mathcal{X}_i$ corresponding to a maximal chamber of the secondary fan of X (Corollary 8.3). This suggests that the HHL resolution should somehow be an invariant of the Cox category, and the goal of this subsection is to make this intuition precise by constructing a "resolution of the diagonal for the Cox category" that pushes forward to the HHL resolutions of the stacks $\mathcal{X}_i$.

We begin with an auxiliary construction. Recall that each X_i is of the form $(\mathrm{Spec}(S) \setminus V(B_i))/G$ for some monomial ideal $B_i \subseteq S$ and an action of a group G arising from the $\mathrm{Cl}(X)$-grading on S. Let $I \coloneqq \sum B_i$, and let $\mathscr{X}$ denote the toric stack $[(\mathrm{Spec}(S) \setminus V(I))/G]$. We want to take an HHL resolution of the diagonal for $\mathscr{X}$. We cannot quite directly do this, because $\mathscr{X}$ need not be separated (i.e. the diagonal need not be a closed substack), and one can only take an HHL resolution of a closed substack. So instead, we consider the HHL resolution of the closure of the diagonal torus in $\mathscr{X} \times \mathscr{X}$. Let H denote the complex of free $S \otimes_{\mathbf{k}} S$-modules associated to this complex of $\mathcal{O}_{\mathscr{X} \times \mathscr{X}}$-modules, and let $\mathcal{H}_{i,j}$ denote the HHL resolution of the (closure of the) graph of the birational map $\mathcal{X}_i \dashrightarrow \mathcal{X}_j$. The following result is a consequence of technical machinery in [HHL24]; we omit the proof.

Proposition 17.4. *For any i, j, $H|_{\mathcal{X}_i \times \mathcal{X}_j}$ is homotopy equivalent to $\mathcal{H}_{i,j}$.*

Applying the functor Ψ from §17.2 to H gives an object $\mathcal{H} \in D_{\mathrm{Cox}}(X \times X)$. Choose a smooth toric stack $\mathcal{X}$ as in the definition of $D_{\mathrm{Cox}}(X)$, so that $D_{\mathrm{Cox}}(X)$ is a full triangulated subcategory of $D^b(\mathcal{X})$, and let $\pi_i\colon \mathcal{X} \to \mathcal{X}_i$ denote the accompanying morphisms of toric stacks. Of course, $D_{\mathrm{Cox}}(X \times X)$ is a full triangulated subcategory of $D(\mathcal{X} \times \mathcal{X})$ as well. It follows from Proposition 17.4 that the object $(\pi_i \times \pi_j)_* \mathcal{H}$ is homotopy equivalent to $\mathcal{H}_{i,j}$: see [BBB[+]24, Lemma 6.8(1)] for the straightforward proof. Let $\Phi_{i,j}\colon D^b(\mathcal{X}_i) \to D^b(\mathcal{X}_j)$ (resp. $\Phi\colon D^b(\mathcal{X}) \to D^b(\mathcal{X})$) be the Fourier-Mukai transform with kernel $\mathcal{H}_{i,j}$ (resp. $\mathcal{H}$). The functor Φ induces an endofunctor of $D_{\mathrm{Cox}}(X)$. We have a commutative diagram

$$\begin{tikzcd} D(\mathcal{X}_i) \arrow[r, "\Phi_{i,j}"] \arrow[d, "\pi_i^*"'] & D(\mathcal{X}_j) \\ D_{\mathrm{Cox}}(X) \arrow[r, "\Phi"] & D_{\mathrm{Cox}}(X). \arrow[u, "(\pi_j)_*"'] \end{tikzcd} \tag{17.5}$$

To see this, use that pullbacks and pushforwards may be realized as Fourier-Mukai transforms, and refer to the formula for convolution of Fourier-Mukai kernels [Huy06, Proposition 5.10].

Theorem 17.6. *The endofunctor Φ of $D_{\mathrm{Cox}}(X)$ is naturally isomorphic to the identity.*

Proof. One reduces to showing that there is a natural isomorphism $\Phi(\mathcal{O}_{\mathrm{Cox}}(d)) \cong \mathcal{O}_{\mathrm{Cox}}(d)$ for all $d \in \Theta$, and similarly for morphisms of these objects. This proof is straightforward and can probably be suppressed in this talk: see [BBB[+]24, Proof of Corollary 6.10]. □

17.3. **Noncommutative resolution of singularities.**

Definition 17.7. [Kuz08] Let X be a scheme. A categorical (or noncommutative) resolution of singularities for X is a pair of triangulated functors $G\colon \mathcal{D} \to D^b(X)$ and $F\colon \operatorname{Perf}(X) \to \mathcal{D}$, where $\mathcal{D}$ is a smooth triangulated category (see §17.1), such that

(1) We have $\operatorname{Hom}_{\mathcal{D}}(F(\mathcal{F}), D) \cong \operatorname{Hom}_{D^b(X)}(\mathcal{F}, G(D))$ for all $\mathcal{F} \in \operatorname{Perf}(X)$, $D \in \mathcal{D}$.
(2) There is a natural isomorphism $GF \cong \mathrm{id}_{\operatorname{Perf}(X)}$.

Noncommutative resolutions of singularities, in the sense of Van den Bergh [VdB04], give examples of categorical resolutions of singularities: see [Kuz08, §3]. Affine toric varieties are known to admit noncommutative resolutions of singularities [ŠVdB17, FMS19]. The final application of the main result of [BBB$^+$24] that we discuss is a generalization of this result beyond the affine case.

Let X be a semiprojective toric variety (see [CLS11, §7.2] for the definition). For instance, any affine or projective toric variety is semiprojective. Let $G\colon D_{\mathrm{Cox}}(X) \to D^b(X)$ be the pushforward functor, and let $F\colon \operatorname{Perf}(X) \to D_{\mathrm{Cox}}(X)$ be the pullback. Recall that F is fully faithful, and we established in §17.1 that $D_{\mathrm{Cox}}(X)$ is smooth (the smooth and projective assumptions were not necessary for this portion of the argument). Thus, $D_{\mathrm{Cox}}(X)$ is a categorical resolution of singularities for X.

So far, we have not applied the main results of [BBB$^+$24], and we do so now. Let A_Θ be the algebra $\operatorname{End}_{D_{\mathrm{Cox}}(X)}(\bigoplus_{d\in\Theta} \mathcal{O}_{\mathrm{Cox}}(d))$. Since $\bigoplus_{d\in\Theta} \mathcal{O}_{\mathrm{Cox}}(d)$ is a tilting object for $D_{\mathrm{Cox}}(X)$ (this does not require X to be smooth or projective), we have $D_{\mathrm{Cox}}(X) \simeq D^b(A_\Theta)$. We conclude:

Theorem 17.8. *The category $D^b(A_\Theta)$ is a noncommutative resolution of singularities for X. Moreover, A_Θ has global dimension $\dim X$.*

Proof. We have proven the first statement, and the second follows from [FH23, Theorem 2.14]. $\square$

The noncommutative resolutions of singularities for affine toric varieties given by [ŠVdB17, FMS19] follow from Theorem 17.8, though there is just a bit of additional work needed for this: see [BBB$^+$24, Corollary 7.13 and Remark 7.14].

18. Lecture 18 (Matthias Zach): Tate resolutions over projective space

Let $S = \mathbf{k}[x_0, \dots, x_n]$, $\mathbb{Z}$-graded with $\deg(x_i) = 1$, and let $E = \bigwedge_{\mathbf{k}}(e_0, \dots, e_n)$ be the exterior algebra with $\deg(e_i) = -1$. Let E^* denote the $\mathbf{k}$-dual of E. We saw on Monday that there is an equivalence of categories $R\colon D^b(S) \xrightarrow{\simeq} D^b(E)$ that sends a finitely generated module M (thought of as a complex concentrated in cohomological degree 0) to

$$R(M) = \left[\cdots \xleftarrow{\partial_R} E^*(-i) \otimes_{\mathbf{k}} M_i \xleftarrow{\partial_R} E^*(-i+1) \otimes_{\mathbf{k}} M_{i-1} \xleftarrow{\partial_R} \cdots\right], \tag{18.1}$$

where $E^*(-i) \otimes_{\mathbf{k}} M_i$ is in cohomological degree i. Moreover, we saw that we get an induced equivalence

$$\overline{R}\colon D^b(\mathbb{P}^n) \xrightarrow{\simeq} D_{sg}(E) \coloneqq D^b(E)/\operatorname{Perf}(E), \tag{18.2}$$

where $D_{sg}(E)$ denotes the singularity category of E, and $\operatorname{Perf}(E)$ is the category of perfect E-complexes. We recall that the objects in $D_{sg}(E)$ are the same as those in $D^b(E)$, but the morphisms are obtained by inverting all morphisms in $D^b(E)$ whose mapping cone lies in $\operatorname{Perf}(E)$. The equivalence (18.2) gives a "geometric" version of the BGG correspondence, though an unsatisfying one: the passage from $D^b(S)$ to $D^b(\mathbb{P}^n)$ translates to the passage from $D^b(E)$ to $D_{sg}(E)$, which only affects morphisms. Since the morphisms in $D_{sg}(E)$ are difficult to understand concretely, this renders the projective geometry in $D_{sg}(E)$ effectively invisible. This problem is solved by Eisenbud-Fløystad-Schreyer's theory of Tate resolutions [EFS03], and the goal of this lecture is to explain this theory.

Conceptually, we need a way to get our hands on the morphisms in $D_{sg}(E)$. There is a classical method for making the morphisms in a derived category explicit, namely by replacing it with an appropriate homotopy category. For instance, $D^b(E)$ is equivalent to the homotopy category $K^-(E)$ of complexes with finitely generated total cohomology whose terms vanish in small cohomological degrees. In the case of an exterior algebra, this equivalence may be given by taking the $\mathbf{k}$-dual of a minimal free resolution[12]. There is a similar result for the singularity category, due to Buchweitz [Buc21, Theorem 4.4.1], and this is the added ingredient we need. Specifically: there is an equivalence

$$B\colon D_{sg}(E) \xrightarrow{\simeq} K(E), \tag{18.3}$$

where $K(E)$ is the homotopy category of (possibly unbounded on both sides) exact complexes of finitely generated graded free E-modules. This equivalence sends an object C in $D_{sg}(E)$ to the mapping cone of the canonical quasi-isomorphism $G \to F$, where G is the minimal free resolution of the $\mathbf{k}$-dual of C, and F is the $\mathbf{k}$-dual of the minimal free resolution of C. We arrive at the following key commutative diagram:

$$\begin{tikzcd} D^b(S) \arrow[r, "R", "\simeq"'] \arrow[d] & D^b(E) \arrow[r, "\simeq"'] \arrow[d] & K^-(E) \arrow[d] \\ D^b(\mathbb{P}^n) \arrow[r, "\overline{R}", "\simeq"'] & D_{sg}(E) \arrow[r, "B", "\simeq"'] & K(E). \end{tikzcd} \tag{18.4}$$

The only functor here we have not yet described is the rightmost vertical one, which sends an object F to $\operatorname{cone}(G \xrightarrow{\simeq} F)$, where G is the minimal free resolution of F.

With all this in mind, let us now give an alternative formulation of the BGG correspondence for $\mathbb{P}^n$. Essentially, it is given by composing the functors in the bottom row of Diagram (18.4). To obtain an explicit and geometrically meaningful formula for this composition, we proceed as

[12] It makes sense to take a (minimal) free resolution of any complex of finitely generated E-modules with bounded homology [Rob98, Proposition 4.4.1].

follows. Let $\mathcal{F}$ be a coherent sheaf on $\mathbb{P}^n$, and let M be a finitely generated graded S-module such that $\widetilde{M} = \mathcal{F}$. Fix $s > \operatorname{reg}(M)$, the Castelnuovo-Mumford regularity of M. Let G be the minimal free resolution of $R(M_{\geq s})$, and let $T(\mathcal{F})$ be the mapping cone of the quasi-isomorphism $G \xrightarrow{\simeq} R(M_{\geq s})$. Since $T(\mathcal{F})$ is the mapping cone of a quasi-isomorphism, it is exact. Since any two finitely generated graded modules representing $\mathcal{F}$ are isomorphic in large degrees, $T(\mathcal{F})$ is well-defined up to the choices of M and s, up to chain isomorphism.

Definition 18.5 ([EFS03, Section 4])**.** The complex $T(\mathcal{F})$ is called the *Tate resolution of $\mathcal{F}$.*

Example 18.6. Say $S = \mathbf{k}[x_0, x_1]$. Take $M = S/(x_0)$, and $\mathcal{F} = \widetilde{M}$, the structure sheaf of a point on $\mathbb{P}^1$. The regularity of M is zero. We have

$$R(M_{>0}) = \left[\cdots \xleftarrow{e_1} E^*(-2) \xleftarrow{e_1} E^*(-1) \leftarrow 0\right].$$

This complex is exact except in cohomological degree 1. So to get the Tate resolution, we just need to resolve its first cohomology, which gives:

$$T(\mathcal{F}) = \left[\cdots \xleftarrow{e_1} E^*(-1) \xleftarrow{e_1} E^* \xleftarrow{e_1} E^*(1) \xleftarrow{e_1} \cdots\right].$$

The next result demonstrates that the Tate resolution functor is a model for the BGG correspondence $D^b(\mathbb{P}^n) \xrightarrow{\simeq} K(E)$ applied to sheaves.

Proposition 18.7. *For any coherent sheaf $\mathcal{F}$ on $\mathbb{P}^n$, we have $T(\mathcal{F}) \cong (B \circ \overline{R})(\mathcal{F})$.*

Proof. This follows from the commutativity of Diagram (18.4): the formula for T lifts $\mathcal{F}$ to $D^b(S)$ and then applies the top two horizontal and rightmost vertical functors in the diagram. $\square$

In Example 18.6, we only needed to resolve a single module, not a complex with nonzero homology in multiple positions. An advantage to using the truncation $M_{\geq s}$ in the definition of the Tate resolution is that this always happens: $R(M_{\geq s})$ has nonzero cohomology only in degree s, so G is just the minimal free resolution of $H^s(R(M_{\geq s}))$. Let us give a quick sketch of a proof of this fact, as we will need it later. We will use the following technical lemma:

Proposition 18.8. *If M is a finitely generated graded S-module that is r-regular, then $R(M_{\geq r})$ has nonzero cohomology only in degree r.*

Proof. Since $M_{\geq r}$ has a linear free resolution [EG84], this follows immediately from Lemma 3.4. $\square$

The following is one of the main theorems of [EFS03]:

Theorem 18.9. *Let $\mathcal{F}$ be a coherent sheaf on $\mathbb{P}^n$. Its Tate resolution $T(\mathcal{F})$ is minimal, and for all $i \in \mathbb{Z}$, there is an isomorphism $T(\mathcal{F})^i \cong \bigoplus_{j\in\mathbb{Z}} E^*(j-i) \otimes_{\mathbf{k}} H^j(\mathbb{P}^n, \mathcal{F}(-i-j))$.*

Theorem 18.9 tells us how to actually see projective geometry via BGG: the terms of the Tate resolution of $\mathcal{F}$ encode the sheaf cohomology of all the twists of $\mathcal{F}$. That is, the Buchweitz equivalence (18.3) pulls the geometry out of the morphisms in the singularity category and expresses it via the objects of $K(E)$.

Example 18.10. Let us return to Example 18.6. Since $\mathcal{F}$ is the structure sheaf of a point, we have

$$H^0(\mathbb{P}^1, \mathcal{F}(i)) = 1 \quad \text{for all } i, \quad H^j(\mathbb{P}^1, \mathcal{F}(i)) = 0 \quad \text{for all } i \text{ and all } j \neq 0$$

One sees this immediately from the terms of $T(\mathcal{F})$ using Theorem 18.9.

Remark 18.11. Let $\mathcal{F}$ be a sheaf on $\mathbb{P}^n$ and M an S-module such that $\widetilde{M} = \mathcal{F}$. If we defined the Tate resolution of $\mathcal{F}$ without truncating past the regularity of M, Theorem 18.9 would become false. For instance, take $\mathcal{F} = 0$. Of course, we have $\widetilde{\mathbf{k}} = \mathcal{F}$. If we applied the formula for the Tate resolution without truncating $\mathbf{k}$, we would get the complex $E \xleftarrow{1} E$. This is not a minimal complex, and the ranks of its terms clearly do not correspond to the ranks of the cohomologies of the twists of $\mathcal{F}$, which are all zero.

Proof of Theorem 18.9. We will sketch a proof of the formula for the terms of $T(\mathcal{F})$ in greater generality in the next lecture. Concerning minimality: let M be a finitely generated graded S-module such that $\widetilde{M} = \mathcal{F}$, and let $s > \operatorname{reg}(M)$. Recall that $R(M_{\geq s})$ has cohomology concentrated in degree s [EFS03, Corollary 2.4]. Let $\mathfrak{m}_E$ be the homogeneous maximal ideal of E. The complex G is the minimal free resolution of the kernel of the first nonzero differential in the minimal complex $R(M_{\geq s})$. So we just need to check that the quasi-isomorphism $G \xrightarrow{\simeq} R(M_{\geq s})$ takes values in $\mathfrak{m}_E \cdot R(M_{\geq s})^s = \mathfrak{m}_E(-s-n-1) \otimes_{\mathbf{k}} M_s$. Equivalently, we must show that the kernel of the first nonzero differential in $R(M_{\geq s})$ is contained in $\mathfrak{m}_E(-s-n-1) \otimes_{\mathbf{k}} M_s$. This holds since $R(M_{\geq s})$ is exact in cohomological degrees greater than s. □

Remark 18.12. The differentials in Tate resolutions, while readily computable in Macaulay2, remain somewhat mysterious. That is, it is not known how to describe them geometrically in general, though progress on this has been made in some special cases, e.g. [Cox07, CM08].

Tate resolutions have had a host of applications; let us highlight two of them. The first is an effective method for computing a Beilinson monad of a sheaf on $\mathbb{P}^n$, by which we mean a complex of the form

$$(18.13)\quad \cdots \leftarrow \bigoplus_{i=0}^{n} H^j(\mathcal{F}(j-i+1)) \otimes_{\mathbf{k}} \Omega^{i-j-1}_{\mathbb{P}^n}(i-j-1) \leftarrow \bigoplus_{i=0}^{n} H^i(\mathcal{F}(j-i)) \otimes_{\mathbf{k}} \Omega^{i-j}_{\mathbb{P}^n}(i-j) \leftarrow \cdots,$$

that is quasi-isomorphic to the sheaf $\mathcal{F}$, where the rightmost term is in cohomological degree j. This complex is typically obtained via a Fourier-Mukai transform with kernel given by the Beilinson resolution of the diagonal, but, from this point of view, it is not easy to compute the differentials in (18.13), even with the aid of Macaulay2. The insight of [EFS03, Section 6] is that one can first compute the Tate resolution $T(\mathcal{F})$ (which is easy and efficient to do in Macaulay2) and then apply a simple, computable functor to $T(\mathcal{F})$ to obtain the monad (18.13). These ideas were also applied in Eisenbud-Schreyer's subsequent work on Chow forms [ES03].

The second application is an algorithm for computing sheaf cohomology over $\mathbb{P}^n$ by applying the formula in Theorem 18.9. This is, in some cases, the fastest known way to compute cohomology over $\mathbb{P}^n$ [DE02]. What makes this algorithm fast is that it involves taking minimal free resolutions over an exterior algebra: since exterior algebras are finite dimensional over a field, homological calculations over them are particularly fast.

19. Lecture 19 (Sasha Zotine): Tate resolutions over toric varieties

Eisenbud-Fløystad-Schreyer's theory of Tate resolutions [EFS03] was extended to products of projective spaces in [EES15] and later generalized further to toric varieties in [BE26, BE]. The goal of this lecture is to give an overview of some of the results in these latter papers.

Let $\mathcal{X}$ be the toric stack associated to a simplicial projective toric variety X. As discussed earlier, if X is smooth, then $\mathcal{X} = X$. Let $S = \mathbf{k}[x_0, \ldots, x_n]$ be the Cox ring of X, equipped with its $\operatorname{Cl}(X)$-grading. To ease notation, we write $A := \operatorname{Cl}(X)$ in this section. To generalize Tate resolutions to sheaves on $\mathcal{X}$, the first thing we need to do is extend the BGG correspondence to A-graded polynomial rings. This already introduces a complication: in the standard graded case, the internal grading on a module M over the polynomial ring induces the homological grading on the complex $R(M)$. This cannot possibly work when the A-grading is nonstandard. The solution to this problem is to widen our perspective from complexes of exterior modules to *differential modules*.

Before explaining what we mean, we fix some notation. Let $E = \bigwedge_{\mathbf{k}}(e_0, \ldots, e_n)$, equipped with the $A \times \mathbb{Z}$-grading such that $\deg(e_i) = (-\deg(x_i); -1)$.

Definition 19.1. A *differential E-module* is an $A \oplus \mathbb{Z}$-graded right E-module D equipped with an E-linear endomorphism ∂_D of degree $(0; -1)$ such that $\partial_D^2 = 0$. A *morphism* of these objects is the obvious thing: an E-linear degree $(0; 0)$ map that commutes with differentials. Homotopies of such morphisms are also defined in the natural way: see [BE26, Section 2] for details.

Remark 19.2. We can think of E as a differential graded (dg) algebra with cohomological grading given by $\deg(e_i) = 1$, internal grading given by $\deg(e_i) = -\deg(x_i)$, and trivial differential. The category of dg-modules over this dg-algebra is equivalent to $\mathrm{DM}(E)$. An advantage to repackaging this data as a differential E-module is that it makes computation in Macaulay2 a bit simpler.

Let $\mathrm{mod}(S)$ be the category of A-graded S-modules, and let $\mathrm{DM}(E)$ be the category of differential E-modules. The A-graded BGG functor

$$R\colon\ \mathrm{mod}(S) \to \mathrm{DM}(E)$$

sends $M \in \mathrm{mod}(S)$ to $\bigoplus_{a \in A} E^*(-a; 0) \otimes_{\mathbf{k}} M_a$, with differential given by multiplication on the left by $\sum_{i=0}^n e_i \otimes x_i$. The functor R extends naturally to complexes of A-graded S-modules, and it has a left adjoint, just as in the classical setting. In fact, the functor R induces an equivalence $D^b(S) \xrightarrow{\simeq} D^b_{\mathrm{DM}}(E)$, where the target is the bounded derived category of differential E-modules, i.e. the category of differential E-modules with finitely generated homology with quasi-isomorphisms inverted. But we will not need this.

Remark 19.3. When S is standard graded, there is an equivalence between $\mathrm{DM}(E)$ and the category of complexes of $\mathbb{Z}$-graded E-modules, where $\deg(e_i) = -1$. The BGG functor we define above coincides with the classical one via this equivalence.

Example 19.4 ([BE25, Example 3.3])**.** Assume $\mathrm{char}(\mathbf{k}) \neq 2$, and let us consider the case of the weighted projective stack $\mathbb{P}(1, 1, 2)$, so that $\mathrm{Cl}(X) = \mathbb{Z}$, and $S = k[x_0, x_1, x_2]$ with $\deg(x_0) = 1 = \deg(x_1)$ and $\deg(x_2) = 2$. Let $M = S/(x_0^4 + x_1^4 + x_2^2)$ be the coordinate ring of a genus 1 curve in $\mathbb{P}(1, 1, 2)$. The differential module $R(M)$ is an infinitely generated free E-module whose generators are in bijection with a $\mathbf{k}$-basis for M. In this example, we have $\dim_{\mathbf{k}} M_0 = 1$, and $\dim_{\mathbf{k}} M_i = 2i$ for $i \geq 1$. Thus, $R(M)$ looks like:

$$\begin{tikzcd}
R(M) = E^* \arrow[r] \arrow[rr, bend right] & E^*(-1;0)^{\oplus 2} \arrow[r] \arrow[rr, bend left] & E^*(-2;0)^{\oplus 4} \arrow[r] \arrow[rr, bend right] & E^*(-3;0)^{\oplus 6} \arrow[r] & \cdots
\end{tikzcd}$$

The above graphic is an attempt to represent a differential module, not a complex. To be more precise, we choose monomial bases for each M_i, e.g. $\{1\}, \{x_0, x_1\}$ and $\{x_0^2, x_0x_1, x_1^2, x_2\}$ for M_0, M_1 and M_2, respectively. The differential ∂ sends the generator $1 \in E^*$ to $e_0 \otimes x_0 + e_1 \otimes x_1 + e_2 \otimes x_2$. This element lies in the direct sum $E^*(-1; 0)^{\oplus 2} \oplus E^*(-2; 0)^{\oplus 4}$. This is indicated in the above graphic by the arrows emanating from E^*.

Let us now turn to Tate resolutions. We want an equivalence between $D^b(\mathcal{X})$ and some homotopy category of differential E-modules. Moreover, we want a formula for this equivalence that is geometrically meaningful, which is what the theory of Tate resolutions over $\mathbb{P}^n$ provides (see Theorem 18.9).

Naively generalizing the formula for Tate resolutions from the previous lecture doesn't work: let us explain why. Recall that one defines the Tate resolution of a sheaf $\mathcal{F}$ on $\mathbb{P}^n$ by choosing a graded module M such that $\widetilde{M} = \mathcal{F}$, taking the minimal free resolution G of $R(M_{\geq s})$ where $s > \operatorname{reg}(M)$, and setting $T(\mathcal{F}) = \operatorname{cone}(G \xrightarrow{\simeq} R(M_{\geq s}))$. This recipe can't possibly work in the toric case, because the entire construction only depends on the Cox ring S: the irrelevant ideal never enters in. It therefore cannot fully capture $D^b(\mathcal{X})$.

We therefore proceed as follows. The maximal cones of Σ determine a G-invariant affine open cover of $\mathbb{A}^{n+1} \setminus V(B)$, where B is the irrelevant ideal of X [BCS05, §4]. Given a complex $\mathcal{F}$ of quasi-coherent $\mathcal{O}_{\mathcal{X}}$-modules, we let $\mathcal{C}(\mathcal{F})$ denote the Čech complex of $\mathcal{F}$ with respect to this open cover, and we let $\mathcal{C}_*(\mathcal{F})$ be the complex $\bigoplus_{a \in \operatorname{Cl}(X)} \mathcal{C}(\mathcal{F}(a))$ of $\operatorname{Cl}(X)$-graded S-modules. Let $\operatorname{coh}(\mathcal{X})$ (resp. $\operatorname{Com}(S)$) denote the category of coherent sheaves on $\mathcal{X}$ (resp. complexes of A-graded S-modules), and let $\mathbf{R}\Gamma_* \colon \operatorname{coh}(\mathcal{X}) \to \operatorname{Com}(S)$ denote the functor given by $\mathcal{F} \mapsto \mathcal{C}_*(\mathcal{F})$.

Example 19.5. Let X be the Hirzebruch surface of type r. In this case, the canonical affine open cover is given by the open sets $\operatorname{Spec}(S) \setminus V(m)$, where m ranges over the set of monomials $M = \{x_0x_1, x_0x_3, x_1x_2, x_2x_3\}$. The complex $\mathcal{C}_*(\mathcal{O}_X)$ therefore has the form

$$S[\tfrac{1}{x_0x_1x_2x_3}] \leftarrow S[\tfrac{1}{x_0x_1x_2x_3}]^{\oplus 4} \leftarrow \Big(\bigoplus_{i=0}^{3} S[\tfrac{1}{x_0 \cdots \widehat{x_i} \cdots x_3}]\Big) \oplus S[\tfrac{1}{x_0x_1x_2x_3}]^{\oplus 2} \leftarrow \bigoplus_{m \in M} S[\tfrac{1}{m}],$$

with differential given by the usual signed localization maps.

We define $\Phi \colon \operatorname{coh}(\mathcal{X}) \to \operatorname{DM}(E)$ to be the composition $\mathbf{R} \circ \mathbf{R}\Gamma_*$. The toric Tate resolution of a sheaf $\mathcal{F}$ on $\mathcal{X}$ is defined to be the minimization of $\Phi(\mathcal{F})$. More specifically, we have the following, which is implied by [BE26, Theorem 3.3]:

Theorem 19.6. *Let $\mathcal{F} \in \operatorname{coh}(\mathcal{X})$. There exists $T(\mathcal{F}) \in \operatorname{DM}(E)$ with the following properties:*

(1) $T(\mathcal{F})$ is homotopy equivalent to $\Phi(\mathcal{F})$.
(2) $T(\mathcal{F})$ is free as an E-module, and it is minimal, i.e. its differential takes values in $\mathfrak{m}_E \cdot T(\mathcal{F})$, where $\mathfrak{m}_E$ is the homogeneous maximal ideal of E.
(3) We have $T(\mathcal{F}) \cong \bigoplus_{(a;j) \in A \oplus \mathbb{Z}} H^j(\mathcal{X}, \mathcal{F}(a)) \otimes_{\mathbf{k}} E^*(-a;j)$.
(4) $T(\mathcal{F})$ is exact.

Moreover, $T(\mathcal{F})$ is unique up to isomorphism of differential E-modules.

Proof. We can think of $\Phi(\mathcal{F})$ as a 1-periodic complex $\Big[\cdots \leftarrow \Phi(\mathcal{F})(0;-1) \xleftarrow{\partial_{\Phi(\mathcal{F})}} \Phi(\mathcal{F}) \leftarrow \cdots\Big]$. This complex is the totalization of the following bicomplex:

$$\begin{tikzcd}
 & \vdots \arrow[d] & & \vdots \arrow[d] & \\
\cdots & \bigoplus_{a\in\operatorname{Cl}(X)} \mathcal{C}(\mathcal{F}(a))^j \otimes_{\mathbf{k}} E^*(-a;i-1) \arrow[l] \arrow[d, "d^{\text{Čech}}\otimes 1"] & & \bigoplus_{a\in\operatorname{Cl}(X)} \mathcal{C}(\mathcal{F}(a))^j \otimes_{\mathbf{k}} E^*(-a;i) \arrow[ll, "\sum_{i=0}^n x_i\otimes e_i"'] \arrow[d, "d^{\text{Čech}}\otimes 1"] & \cdots \arrow[l] \\
\cdots & \bigoplus_{a\in\operatorname{Cl}(X)} \mathcal{C}(\mathcal{F}(a))^{j+1} \otimes_{\mathbf{k}} E^*(-a;i-1) \arrow[l] \arrow[d] & & \bigoplus_{a\in\operatorname{Cl}(X)} \mathcal{C}(\mathcal{F}(a))^{j+1} \otimes_{\mathbf{k}} E^*(-a;i) \arrow[ll, "-\sum_{i=0}^n x_i\otimes e_i"'] \arrow[d] & \cdots \arrow[l] \\
 & \vdots & & \vdots &
\end{tikzcd}$$

(where the $(-i,j)$ position is the top right term). We can use [EFS03, Lemma 3.5] to minimize the totalization of this bicomplex, and parts (1) - (3) follow immediately from this result. The uniqueness of $T(\mathcal{F})$ follows from minimality, via the usual Nakayama's Lemma argument. Part (4) holds since the rows of the bicomplex are exact. To see this, we start by setting

$$K := \left[\cdots \leftarrow \bigoplus_{a \in \mathrm{Cl}(X)} S(a) \otimes_{\mathbf{k}} E^*(-a; i-1) \xrightarrow{\sum_{i=0}^n x_i \otimes e_i} \bigoplus_{a \in \mathrm{Cl}(X)} S(a) \otimes_{\mathbf{k}} E^*(-a; i) \leftarrow \cdots \right].$$

As a complex of S-modules, this is just $\bigoplus_{a \in \mathrm{Cl}(X), i \in \mathbb{Z}} \mathrm{Kos}(x_0, \dots, x_n)(a)[i]$, where $\mathrm{Kos}(x_0, \dots, x_n)$ is the Koszul complex on the variables of S. Let M be a module such that $\widetilde{M} = \mathcal{F}$. The homology of the complex $M \otimes_S K$ is annihilated by the maximal ideal of S; it follows that each row in the above bicomplex is exact. $\square$

Definition 19.7. The *Tate resolution* of $\mathcal{F} \in \mathrm{coh}(\mathcal{X})$ is the differential E-module $T(\mathcal{F})$ in Theorem 19.6.

Remark 19.8. If $\mathcal{F}$ is a coherent sheaf on $\mathcal{X}$, and M is a $\mathrm{Cl}(X)$-graded S-module such that $\widetilde{M} = \mathcal{F}$ and $H^0_B(M) = 0$ (where B is the irrelevant ideal of X), there is an injection $\mathbf{R}(M) \hookrightarrow T(\mathcal{F})$. When $X = \mathbb{P}^n$, Remark 19.3 implies that $T(\mathcal{F})$ may be considered as a complex of exterior modules, and one can show that it is the minimal free resolution of this embedded copy of $\mathbf{R}(M)$. This demonstrates that Definition 19.7 recovers the classical definition over $\mathbb{P}^n$.

In fact, Tate resolutions over weighted projective stacks can also be recast in terms of minimal free resolutions, in a manner similar to Tate resolutions over $\mathbb{P}^n$: see [BE26, Theorem 3.7]. This allows us to easily generate examples of Tate resolutions over weighted projective stacks: if the speaker wishes to discuss some, we refer to [BE26, Section 3.4]. This leads to an algorithm for computing sheaf cohomology over weighted projective stacks (and schemes) [BE25]. When $\mathrm{Cl}(X)$ has rank greater than 1, examples of Tate resolutions are more difficult to write down explicitly: see e.g. [BE26, Example 3.14] for a schematic of a Tate resolution over a Hirzebruch surface. It is unknown whether toric Tate resolutions can be constructed using free resolutions in general.

We now have a theory of toric Tate resolutions and a formula for their underlying modules that generalizes Theorem 18.9. But we have not yet explained how toric Tate resolutions induce an equivalence between $D^b(\mathcal{X})$ and a homotopy category of differential E-modules. To do so, we must recall an interesting property of toric Tate resolutions called X-exactness.

Definition 19.9. Given $I \subseteq \{0, \dots, n\}$, we say I is irrelevant if the ideal $(x_i : i \in I) \subseteq S$ contains the irrelevant ideal B of X. Let Q_I denote the ideal $(e_i : i \notin I) \subseteq E$. We say $D \in \mathrm{DM}(E)$ is X-exact if $D \otimes_E E/Q_I$ is exact whenever I is irrelevant.

Remark 19.10. The minimal irrelevant sets I are in natural bijection with the primitive collections of X.

Example 19.11. If $X = \mathbb{P}^n$, then I is irrelevant if and only if $I = \{0, \dots, n\}$. In this case, $Q_I = 0$, and so $E/Q_I = E$. Thus, $\mathbb{P}^n$-exactness is the same as exactness.

Proposition 19.12. *If $\mathcal{F} \in \mathrm{coh}(\mathcal{X})$, then its toric Tate resolution $T(\mathcal{F})$ is X-exact.*

Proof. Two homotopy equivalent differential E-modules are X-exact simultaneously, so it suffices to show $\Phi(\mathcal{F})$ is X-exact. This follows from the proof of Theorem 19.6, since the rows of the bicomplex in that proof, considered as differential E-modules, are X-exact. $\square$

Let $K_{\mathrm{DM}}^{X\text{-exact}}(E)$ denote the homotopy category of free differential E-modules F that are X-exact, and such that $\dim_{\mathbf{k}} \bigoplus_{i \in \mathbb{Z}} F_{(a;i)}$ is finite dimensional for all $a \in A$. The following theorem, which will appear in forthcoming work of Brown-Erman [BE], gives the desired equivalence of categories between $D^b(X)$ and a homotopy category of differential E-modules, generalizing the geometric BGG correspondence from the last lecture from $\mathbb{P}^n$ to toric stacks.

Theorem 19.13. *The toric Tate resolution functor extends to a map $T\colon D^b(\mathcal{X}) \to K_{\mathrm{DM}}^{X\text{-}exact}(E)$, and it is an equivalence of categories.*

Sketch of part of a proof. All we have shown is that the Tate resolution of a sheaf lies in $K_{\mathrm{DM}}^{X\text{-exact}}(E)$. Showing that our Tate resolution functor extends to a fully faithful map $D^b(\mathcal{X}) \to K_{\mathrm{DM}}^{X\text{-exact}}(E)$ is not much more work, but showing that this functor is essentially surjective is beyond the scope of these notes. $\square$

Appendix A. Using the Macaulay2 Package

The Macaulay2 code is currently contained in the `HHLResolutions` package. As this is not a package distributed with Macaulay2 using it is slightly harder than usual. In particular you need to be aware of where the code is actually located on your computer. This tutorial will assume that you have Macaulay2 installed and have some familiarity with using it.

A.1. Basic Usage and Installation.

The most direct way is to call `needsPackage` with the file name specified. For instance if the file `HHLResolutions.m2` is in the folder `/home/username/code/`, then one can run the following to load the package.

```
needsPackage("HHLResolutions", FileName=>"/home/username/code/HHLResolutions.m2")
```

Then we can use this for example to compute a resolution for the diagonal in $\mathbb{P}^2 \times \mathbb{P}^2$.

```
needsPackage("HHLResolutions", FileName=>"/home/username/code/HHLResolutions.m2")
X = toricProjectiveSpace 2;
Y = X ** X; -- This is the cartesian product X \times X
phi = diagonalToricMap(X); -- This is the diagonal map from X into X \times X
hhlResolution(phi)
```

The documentation of the package is viewable by using viewHelp

```
needsPackage("HHLResolutions", FileName=>"/home/username/code/HHLResolutions.m2")
viewHelp HHLResolutions
```

If you wish to install the package for your version of Macaulay2, then you can run the following. It would need to be run every time the package is updated, however after the package is installed, one can run needsPackage from anywhere without worrying about the `FileName` parameter.

```
--This needs to be run from the folder containing the HHLResolutions.m2 file
installPackage("HHLResolutions")
```

You can also provide a FileName parameter to installPackage if you wish. Installing the package is **NOT** required.

A.2. Computing Resolutions.

To get a resolution for a toric subvariety, you should use the `hhlResolution` function. This function currently supports two ways of being called, First is to give it a toric map, represented by a `ToricMap` object as seen previously.

For example, we can recover the examples from Section 7 as follows

```
X = affineSpace 2
Y = affineSpace 1
S = ring X
f = map(X,Y,map(ZZ^2,ZZ^1,matrix {{1},{1}}))
hhlResolution(f)
g= map(X,Y,map(ZZ^2,ZZ^1,matrix {{2},{3}}))
hhlResolution(g)
```

For the resolution of a point, NormalToricVarieties will not allow maps to and from a single point, so we need to use the alternate format

```
X = hirzebruchSurface 3
S = ring X
hhlResolution(X,map(ZZ^(dim X),ZZ^0,0))
```

A.3. Bondal Thomsen Monads of Line Bundles.

There is also support for obtaining the result of Fourier-Mukai of line bundles against the HHL resolution of the diagonal.

```
X = hirzebruchSurface 3
S = ring X -- the ring has variables x_0,x_1,x_2,x_3
a = {0,0,2,-1} --this is the exponent vector of the monomial x_2^2/x_3
lineBundleBondalThomsenMonad(X,a)
```

A.4. **Other Functions of Note.** There are a few other functions which may potentially be of interest, although in most cases the functions previously described suffices for accessing the functionality of the package.

- `hhlVectors`**:** The cell complex supporting the resolution is cut out by hyperplanes, this returns the normals of those hyperplanes
- `bondalThomsenStrata`**:** This function returns the cells on a fundalmental domain that correspond to the Bondal-Thomsen strata grouped by stratum.
- `hhlPolytopes`**:** This function returns the cells on a fundamental domain that are used to construct the resolution. This is primarily intended for internal use so the output is somewhat inscrutible. See Section 7 for the construction.

There are other functions, however nearly all of the rest are either not relevant to the questions posed here or are intended to expose certain internal details.

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