MATH1627, Dr. Smith, Test #2 Tuesday Oct. 26, 2021.

Make sure to show all your work. You may not receive full credit if the accompanying work is incomplete or incorrect. If you do scratch work make sure to indicate scratch work - I will not take off points for errors in the scratch work if it is so labeled and will assume that the scratch work is not part of the final answer. You may omit one problem (if you do all 8 I will base your grade on the best 7 out of 8.)

I. Just set up the integrals needed to calculate the indicated quantities. For full credit you must have the correct limits of integration and the correct variable with respect to which the function is to be integrated.

Problem. 1. Find the centroid (center of mass) of the finite region in the plane bounded by $y = 3x$ and $y = x^2$.

Solution.

$$\begin{aligned} \overline{x} &= \int_0^3 x(3 - x^2)dx \Big/ \int_0^3 3x - x^2 dx \\ \overline{y} &= \int_0^3 \frac{1}{2}(9x^2 - x^4)dx \Big/ \int_0^3 3x - x^2 dx. \end{aligned}$$

□

Problem. 2. Find the surface area of the sheet obtained by rotating (a full 360 degrees) the portion of $y = \sqrt[3]{x}$ between $x = 0$ and $x = 8$ around

a.) the x-axis;

Solution.

$$\int_0^8 2\pi x^{\frac{1}{3}} \sqrt{1 + \frac{1}{9}x^{-\frac{4}{3}}}dx$$

□

b.) the y-axis.

Solution.

$$\int_0^2 2\pi y^3 \sqrt{1 + 9y^4}dy$$

□

II. Determine if the following series converge, indicate your reasoning and indicate which test you are using.

3. $\displaystyle\sum_{n=1}^{\infty} \frac{3n-5}{\sqrt{n^5+7n}}$ 4. $\displaystyle\sum_{n=1}^{\infty} \frac{1\cdot 4\cdot 7\cdot \ldots \cdot (3n-2)}{n!2^n}$

5. $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{\ln n}{\sqrt{n}}$ 6. $\displaystyle\sum_{n=1}^{\infty} \frac{(2n)!}{n!\ n^n}$

Solution. # 3: Use the limit comparison test with $\sum \frac{1}{x^{\frac{3}{2}}}$; the limit is 3 so the series converges. (For straight comparison - the inequality is only valid for $n > 1$.) □

Solution. # 4: Use the ratio test; the limit of the ratio is $\frac{3}{2}$ so the series diverges. □

Solution. # 5: Use the alternating series test:
a.) the series is clearly alternating.
b.)

$$\lim_{n\to\infty} \frac{\ln(n)}{\sqrt{n}} = \lim_{n\to\infty} \frac{1/n)}{1/(2\sqrt{n})} = \lim_{n\to\infty} \frac{2}{\sqrt{n}} = 0$$

So the limit of the terms is 0.
c.)

$$\begin{aligned} f(x) &= \frac{\ln(x)}{\sqrt{x}} \\ f'(x) &= \frac{\frac{1}{x}\sqrt{x} - \ln x \frac{1}{2\sqrt{x}}}{x} = \frac{2-\ln x}{2x\sqrt{x}} \end{aligned}$$

which is negative for $x > e^2$; so the function is eventually decreasing. Which means the series converges by the alternating series test. □

Solution. # 6: Use the ratio test; the limit of the ratio is $\frac{4}{e}$ so the series diverges. □

II. Problems 7 and 8. Show your work and indicate your reasoning:

Problem. 7. Argue that the following series converges and estimate how many terms of the series are necessary for the sum of those terms to be within 0.001 of the value of the series:

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt[3]{n}}.$$

Solution. We want $|a_{n+1}| < .001$ (but I also accepted $|a_n| < .001$.)

$$\begin{aligned} \frac{1}{\sqrt[3]{n+1}} &< .001 \\ \sqrt[3]{n+1} &> 1000 \\ n &> 1000^3 - 1. \end{aligned}$$

or

$$\begin{aligned} \frac{1}{\sqrt[3]{n}} &< .001 \\ \sqrt[3]{n} &> 1000 \\ n &> 1000^3. \end{aligned}$$

$\square$

Problem. 8. Argue that the following series converges for all $p \geq 1$:

$$\sum_{n=1}^{\infty} \frac{n^p}{n!}.$$

Solution. Using the ratio test:

$$\lim_{n\to\infty} \frac{(n+1)^p}{(n+1)!} \cdot \frac{n!}{n^p} = \lim_{n\to\infty} \left(\frac{n+1}{n}\right)^p \cdot \frac{1}{n+1} = 1^p \cdot 0 = 0.$$

So the series converges for all p. $\square$

Extra credit: A bagel in the shape of a (mathematical) torus has a diameter of 10 inches and its hole has a diameter of 3 inches. Calculate:

a.) the volume of the bagel,

Solution. The volume is the area of a circle of radius $\frac{7}{4}$ times the circumference of a circle of radius $\frac{13}{4}$. $\square$

b.) the surface area of the bagel.

Solution. The surface area is the circumference of a circle of radius $\frac{7}{4}$ times the circumference of a circle of radius $\frac{13}{4}$. $\square$

Give exact values.