MATH2660 Dr. Smith Test 1, July 7, 2023.

Make sure to show all your work. You may not receive full credit if the accompanying work is incomplete or incorrect. If you do scratch work make sure to indicate scratch work - I will not take off points for errors in the scratch work if it is so labeled and will assume that the scratch work is not part of the final answer. You may use your calculators.

Problem 1. Use the Gaussian elimination procedure to solve the following system of equations:

$$\begin{aligned} x - 6y - 2z &= 5 \\ 3x + 2y + 2z &= 1 \\ 2x + 2y + 3z &= -4. \end{aligned}$$

Solution.

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ \frac{1}{2} \\ -3 \end{bmatrix}$$

$\square$

Problem 2 & 3. Find the inverse of the following matrices:

$$(2.) \begin{bmatrix} 3 & -2 \\ 5 & 4 \end{bmatrix} \quad (3.) \begin{bmatrix} 3 & -1 & 0 \\ 5 & 4 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Solution.

$$(2.) \begin{bmatrix} 3 & -2 \\ 5 & 4 \end{bmatrix}^{-1} = \frac{1}{22} \begin{bmatrix} 4 & 2 \\ -5 & 3 \end{bmatrix}$$

$\square$

Solution.

$$(3.) \begin{bmatrix} 3 & -1 & 0 \\ 5 & 4 & 0 \\ 0 & 0 & 1 \end{bmatrix}^{-1} = \frac{1}{17} \begin{bmatrix} 4 & 1 & 0 \\ -5 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$\square$

Problem 4. Find the inverse of the following matrix:

$$\begin{bmatrix} 1 & 1 & 2 \\ 3 & 1 & 0 \\ -2 & 0 & 3 \end{bmatrix}$$

Solution.

$$\begin{bmatrix} 1 & 1 & 2 \\ 3 & 1 & 0 \\ -2 & 0 & 3 \end{bmatrix}^{-1} = \begin{bmatrix} -\frac{3}{2} & \frac{3}{2} & 1 \\ \frac{9}{2} & -\frac{7}{2} & -3 \\ -1 & 1 & 1 \end{bmatrix}$$

□

Problem 4 & 5. Find the determinant of the following matrices:

$$(4.) \begin{bmatrix} 3 & 2 & -2 \\ -1 & 4 & 3 \\ 2 & -2 & -3 \end{bmatrix} \quad (5.) \begin{bmatrix} 3 & 1 & 0 & 2 \\ 4 & 0 & 1 & -1 \\ 0 & 0 & -2 & 3 \\ -1 & 2 & -3 & 0 \end{bmatrix}$$

Solution.

$$\begin{aligned} (4.)\det &= 0 \\ (5.)\det &= -61 \end{aligned}$$

□

Problem 6. Prove that if A is a 2×2 and B is obtained from A by replacing the second row by the sum of the two rows, then they have the same determinants.

Proof. Let:

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

then

$$\det(A) = ad - cb.$$

Let:

$$B = \begin{bmatrix} a & b \\ c+a & d+b \end{bmatrix}.$$

then

$$\begin{aligned} \det(B) &= a(d+b) - (c+a)b \\ &= ad + ab - cb - ab \\ &= ad - cb \\ &= \det(A) \end{aligned}$$

which proves the theorem. $\square$

Problem 7 a.) Argue that the following two matrices are inverses of each other

$$\begin{bmatrix} 1 & 1 & -1 \\ -3 & 2 & -1 \\ 3 & -3 & 2 \end{bmatrix} \quad \begin{bmatrix} 1 & 1 & 1 \\ 3 & 5 & 4 \\ 3 & 6 & 5 \end{bmatrix}$$

Solution. Multiply them out and you'll get the identity matrix. $\square$

Problem 7 b. Use the information from part (a) to solve the following systems of equations:

$$\begin{bmatrix} 1 & 1 & 1 \\ 3 & 5 & 4 \\ 3 & 6 & 5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ -2 \\ 5 \end{bmatrix}$$

Solution.

$$\begin{bmatrix} 1 & 1 & -1 \\ -3 & 2 & -1 \\ 3 & -3 & 2 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 3 & 5 & 4 \\ 3 & 6 & 5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 & 1 & -1 \\ -3 & 2 & -1 \\ 3 & -3 & 2 \end{bmatrix} \begin{bmatrix} 3 \\ -2 \\ 5 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -4 \\ -18 \\ 25 \end{bmatrix}$$

$\square$

Problem 8 Determine if the following sets are linear subspaces of the indicated space S.

a.) The set of vectors $(t, -2t, 3u)$ in $S = \mathbb{R}^3$ where t and u are real numbers.

b.) For the space S of continuous functions on the interval $[a, b]$ the set of functions f so that $f(a) = 0$ and $f(b) = 0$.

c.) The set of invertible matrices from the space S of 3×3 matrices.

d.) The set of vectors (x, y, z) in $S = \mathbb{R}^3$ at least one coordinate of which is zero.

Solution. (a.) and (b.) are vector spaces. (c.) is not because $0 \times M$ for any M is the zero matrix which is not invertible. (d.) is not: consider (e.g.) $(0, 1, 2) + (1, 0, 2) = (1, 2, 4)$. □

Problem 9. Find all the values of k for which the following matrix has a determinant equal to 0:

$$\begin{bmatrix} 3-k & 2 \\ 1 & 2-k \end{bmatrix}$$

Solution.

$$\begin{aligned} (3-k)(2-k) - 2 &= 0 \\ k^2 - 2k - 3k + 6 - 2 &= 0 \\ k^2 - 5k + 4 &= 0 \\ (k-4)(k-1) &= 0 \\ k = 4 \quad &\text{or} \quad k = 1. \end{aligned}$$

□

Problem 10. For each of the values of k obtained for problem 9, find the general solution of the system of equations

$$\begin{bmatrix} 3-k & 2 \\ 1 & 2-k \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

[Hint: Express the solutions in terms of a parameter t.]

Solution. $k = 4$ yields

$$\begin{bmatrix} -1 & 2 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

this gives us

$$-x + 2y \;=\; 0.$$

General solution is

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} t \\ \frac{t}{2} \end{bmatrix};\ t \in \mathbb{R}.$$

$k = 1$ yields

$$\begin{bmatrix} 2 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

this gives us

$$x + y \;=\; 0.$$

General solution is

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} t \\ -t \end{bmatrix};\ t \in \mathbb{R}.$$

□