MATH2660 Dr. Smith Test 2, July 14, 2023.

Please show all your work; you may not receive full credit if the accompanying work is incomplete or incorrect. If you do scratch work make sure to indicate scratch work - I will not take off points for errors in the scratch work if it is so labeled and will assume that the scratch work is not part of the final answer. You may use your calculators.

Problem 1. For the following matrix, find a basis for the row space and a basis for the null space:

$$\begin{bmatrix} 1 & 2 & 3 \\ -2 & 1 & -2 \\ 1 & 7 & 7 \end{bmatrix}$$

Solution.

$$\begin{bmatrix} 1 & 2 & 3 \\ -2 & 1 & -2 \\ 1 & 7 & 7 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 2 & 3 \\ 0 & 5 & 4 \\ 0 & 5 & 4 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 2 & 3 \\ 0 & 5 & 4 \\ 0 & 0 & 0 \end{bmatrix}.$$

Basis for the row space is $\{(1,2,3),(0,5,4)\}$.

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 5 & 4 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x+2y+3z \\ 5y+4z \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

We let z be the parameter t. So

$$\begin{aligned} 5y+4t &= 0 \\ y &= -\frac{4}{5}t \\ x+2y+3z &= 0 \\ x &= -2y-3z \\ &= \frac{8}{5}t-3t \\ (x,y,z) &= \Big(-\frac{7}{5},-\frac{4}{5},1\Big)t \end{aligned}$$

So $(-\frac{7}{5},-\frac{4}{5},1)$ is a basis for the null space.

□

Problem 2. Consider the following set of vectors from $\mathbb{R}^4$:

$$(1,1,0,2),(2,0,1,-1),(2,-4,-3,1).$$

a. Is the set an orthogonal set? Why or why not.
b. Does the set span $\mathbb{R}^4$?
c. Construct an orthonormal set of vectors that spans the same subspace of $\mathbb{R}^4$.

Solution. a. Yes, pairwise their inner products are 0.
b. No, since $\mathbb{R}^4$ has dimension 4 and there are only three vectors.
c.
$$\frac{1}{\sqrt{6}}(1,1,0,2),\frac{1}{\sqrt{6}}(2,0,1,-1),\frac{1}{\sqrt{30}}(2,-4,-3,1).$$

□

Problem 3. Consider the following set of vectors from $\mathbb{R}^3$:

$$(1,1,2),(0,-2,1),(-5,1,2).$$

a. Show that the set is an orthogonal set of vectors.
b. Normalize each of these vectors.
c. Express the vector $(2,4,5)$ as a linear combination of these vectors.

Solution. a. Pairwise their inner products are 0.
b.
$$\frac{1}{\sqrt{6}}(1,1,2),\frac{1}{\sqrt{5}}(0,-2,1),\frac{1}{\sqrt{30}}(-5,1,2).$$
c.
$$(2,4,5) = c_1\frac{1}{\sqrt{6}}(1,1,2)+c_2\frac{1}{\sqrt{5}}(0,-2,1)+c_3\frac{1}{\sqrt{30}}(-5,1,2)$$

Taking the inner product of both sides with each of the orthonormal vectors yields:

$$\begin{aligned}
(2,4,5)\cdot\frac{1}{\sqrt{6}}(1,1,2) &= c_1\\
\frac{16}{\sqrt{6}} &= c_1\\
(2,4,5)\cdot\frac{1}{\sqrt{5}}(0,-2,1)) &= c_2\\
\frac{-3}{\sqrt{5}} &= c_2\\
(2,4,5)\cdot\frac{1}{\sqrt{5}}(-5,1,2)) &= c_3\\
\frac{4}{\sqrt{30}} &= c_3.
\end{aligned}$$

Substituting

$$\begin{aligned}
(2,4,5) &= \frac{16}{6}(1,1,2)+\frac{-3}{5}(0,-2,1)+\frac{4}{30}(-5,1,2)\\
&= \frac{8}{3}(1,1,2)-\frac{3}{5}(0,-2,1)+\frac{2}{15}(-5,1,2).
\end{aligned}$$

□

Problem 4. Determine whether or not the following functions are inner products for $\mathbb{R}^2$: where for $u = (u_1, u_2); v = (v_1, v_1)$ we have,

$$\begin{aligned}
\text{(a.)} \quad < u, v > &= 4u_1v_1 + 3u_2v_2,\\
\text{(b.)} \quad < u, v > &= 4u_1v_1 + 5u_1v_2 + 5u_2v_1 + 3u_2v_2,
\end{aligned}$$

Solution. a. Is an inner product - each of the conditions is satisfied - this needs to be shown.

b. It is not an inner product, the condition that $\langle V, V\rangle \geq 0$ fails; for example if $V = (1, -1)$. □

Problem 5. Consider the following attempts to construct inner products on the linear space of continuous functions over $[0, 2]$. Determine whether or not each is an inner product and,in each case, explain why or why not:

$$\begin{aligned} (\text{a.}) \quad < f, g > &= \int_0^2 f(x)g(x)dx + f(2) \cdot g(2) \\ (\text{b.}) \quad < f, g > &= \int_0^1 f(x)g(x)dx + f(2) \cdot g(2). \end{aligned}$$

[Hint: note that the functions (i.e. vectors) all have domain $[0, 2]$.]

Solution. a. Is an inner product - each of the conditions is satisfied - again this needs to be shown.

b. It is not an inner product, the condition that $< f, f >\geq 0$ only when f is the 0 function fails: let f be any continuous function that's not zero in the open interval $\{x|1 < x < 2\}$. □

Problem 6. For the vectors $v = (2, -1, -2), u = (3, 1, -2)$ calculate:

(a.) the orthogonal projection of u onto v, $\text{proj}_u v$,

(b.) the cosine of the angle between u and v.

Solution.

$$\begin{aligned} \text{a. } \text{proj}_u v &= \frac{u \cdot v}{||v||^2} v = \frac{9}{9}(2, -1, -2) = (2, -1, -2) \\ \text{b. } \quad \cos\theta &= \frac{u}{||u||} \cdot \frac{v}{||v||} = \frac{9}{3\sqrt{14}} = \frac{3}{\sqrt{14}}. \end{aligned}$$

□

Problem 7. Are the following elements of the linear space of polynomials linearly independent?

$$\begin{aligned} f_1(t) &= 1 + t^2 \\ f_2(t) &= t + t^2 \\ f_3(t) &= 1 + t. \end{aligned}$$

Solution.

$$c_1(1+t^2) + c_2(t+t^2) + c_3(1+t) \quad = \quad 0.$$

So

$$\begin{aligned} c_1 + c_3 &= 0 \\ c_2 + c_3 &= 0 \\ c_1 + c_2 &= 0. \end{aligned}$$

$$\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & -2 & 0 \end{bmatrix}$$

The initial 3×3 matrix has non-zero determinant, so the c_i's are zero. So the polynomials are linearly independent. □

Problem 8. Use the Gram-Schmidt process to find an orthonormal set that spams the subspace of $C[-1,1]$ generated by the following elements, use the inner product $\langle f, g \rangle = \int_{-1}^{1} f(x)g(x)dx$:

$$f(x) = 1, g(x) = x^2, h(x) = x^4.$$

Solution.

$$\begin{aligned} w_1 &= f \\ w_2 &= g - \frac{\langle g, w_1 \rangle}{\langle w_1, w_1 \rangle} w_1 \\ w_3 &= h - \frac{\langle h, w_1 \rangle}{\langle w_1, w_1 \rangle} w_1 - \frac{\langle h, w_2 \rangle}{\langle w_2, w_2 \rangle} w_2. \end{aligned}$$

And we have

$$\begin{aligned} \langle g, w_1 \rangle &= \int_{-1}^{1} x^2 dx = \frac{2}{3} \\ \langle w_1, w_1 \rangle &= \int_{-1}^{1} 1 dx = 2 \\ w_2 &= g - \frac{1}{3} = x^2 - \frac{1}{3} \\ \langle w_2, w_2 \rangle &= \int_{-1}^{1} (x^2 - \frac{1}{3})^2 = \frac{8}{45}. \end{aligned}$$

Repeating the process yields:

$$w_3 = x^4 - \frac{6}{7}x^2 + \frac{3}{35}.$$

□

Problem 9. Suppose that the arbitrary 2×2 matrix M has rank 1.:

$$M = \begin{bmatrix} a & b \\ c & d \end{bmatrix},$$

(a.) Show that one row is a multiple of the other.
(b.) Show that the determinant is 0.
(c.) Show that the null space has dimension 1.

Solution. (a.) If it has rank 1 then the row vectors are not linearly independent so there exists non-zero numbers c_1 and c_2 so that

$$\begin{aligned} c_1(a,b) + c_2(c,d) &= 0 \\ (c,d) &= \frac{c_1}{c_2}(a,b). \end{aligned}$$

(b.) Let $k = \frac{c_1}{c_2}$ from part (a.) above. Then

$$M = \begin{bmatrix} a & b \\ ka & kb \end{bmatrix}$$

and $\det M = akb - kab = 0$.

(c.) Since the rank of the matrix is 1, then 1 plus the nullity must equal 2. So the nullity is 1.

For those who didn't use the theorem, here's a proof from the definition,

$$\begin{bmatrix} a & b \\ ka & kb \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

So, to be in the null space we have $ax + by = 0$ and $y = -\frac{a}{b}x$. So the space is spanned by the single vector $[1, -\frac{a}{b}]^T$.

□