MATH2660 Dr. Smith Test 3, July 21, 2023.

Please show all your work; you may not receive full credit if the accompanying work is incomplete or incorrect. If you do scratch work make sure to indicate scratch work - I will not take off points for errors in the scratch work if it is so labeled and will assume that the scratch work is not part of the final answer. You may use your calculators.

Problem 1. Prove that the following transformation $T : \mathbb{R}^3 \to \mathbb{P}_4$ is linear:

$$T(x, y, z) = (2x + 3y)t^4 + (x - y)t^2 - (z - 2x).$$

Solution.

$$\begin{aligned}
T(x_1 + x_2, y_1 + y_2, z_1 + z_2) &= (2(x_1 + x_2) + 3(y_1 + y_2))t^4 \\
&\quad +((x_1 + x_2) - (y_1 + y_2))t^2 \\
&\quad -((z_1 + z_2) - 2(x_1 + x_2)) \\
&= (2x_1 + 3y_1)t^4 + (x_1 - y_1)t^2 - (z_1 - 2x_1) \\
&\quad +(2x_2 + 3y_2)t^4 + (x_2 - y_2)t^2 - (z_2 - 2x_2) \\
&= T(x_1, y_1, z_1) + T(x_2, y_2, z_2).
\end{aligned}$$

And:

$$\begin{aligned}
T(cx, cy, cz) &= (2cx + 3cy)t^4 + (cx - cy)t^2 - (cz - 2cx) \\
&= cT(x, y, z).
\end{aligned}$$

$\square$

Problem 2. Suppose that M is a matrix and $T : \mathbb{R}^3 \to \mathbb{R}^3$ is a linear transformation defined by

$$T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = M \begin{bmatrix} x \\ y \\ z \end{bmatrix}.$$

(a.) Argue that the following vectors span the image of T:

$$T\left(\begin{bmatrix}1\\0\\0\end{bmatrix}\right), T\left(\begin{bmatrix}0\\1\\0\end{bmatrix}\right), T\left(\begin{bmatrix}0\\0\\1\end{bmatrix}\right)$$

(b,) Suppose:

$$T\left(\begin{bmatrix}x\\y\\z\end{bmatrix}\right) = \begin{bmatrix}1 & 2 & 3\\-2 & 1 & -2\\1 & 7 & 7\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}$$

Find a basis for the image of the transformation and a basis for null space.

[Hint: use part (a.).]

Solution. (a.)

$$T\left(x\begin{bmatrix}1\\0\\0\end{bmatrix}+y\begin{bmatrix}0\\1\\0\end{bmatrix}+z\begin{bmatrix}0\\0\\1\end{bmatrix}\right) = T\left(\begin{bmatrix}x\\y\\z\end{bmatrix}\right) = M\begin{bmatrix}x\\y\\z\end{bmatrix}$$

(b.)

$$T\left(\begin{bmatrix}1\\0\\0\end{bmatrix}\right) = \begin{bmatrix}1\\-2\\1\end{bmatrix}; T\left(\begin{bmatrix}0\\1\\0\end{bmatrix}\right) = \begin{bmatrix}2\\1\\7\end{bmatrix}; T\left(\begin{bmatrix}0\\0\\1\end{bmatrix}\right) = \begin{bmatrix}3\\-2\\7\end{bmatrix}$$

So we want a basis for the space spanned by the vectors $(1, -2, 1); (2, 1, 7); (3, -2, 7)$. Using Gaussian elimination:

$$\begin{bmatrix}1 & -2 & 1\\2 & 1 & 7\\3 & -2 & 7\end{bmatrix} \Rightarrow \begin{bmatrix}1 & -2 & 1\\0 & 5 & 5\\0 & 4 & 4\end{bmatrix}$$

So the vectors $(1, -2, 1); (0, 1, 1)$ span the image space.

For the null space we want to solve the system:

$$\begin{bmatrix}1 & 2 & 3\\-2 & 1 & -2\\1 & 7 & 7\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$$

Again using Gaussian elimination

$$\begin{bmatrix} 1 & 2 & 3 \\ -2 & 1 & -2 \\ 1 & 7 & 7 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 2 & 3 \\ 0 & 5 & 4 \\ 0 & 5 & 4 \end{bmatrix}$$

$$\begin{aligned} x + 2y + 3z &= 0 \\ 5y + 4z = 0 \\ z &= -\frac{5}{4}y \\ x &= -2y - 3z \\ &= -\frac{8}{4}y + \frac{15}{4}y = \frac{7}{4}y \end{aligned}$$

So a solution vector is

$$\begin{bmatrix} -\frac{5}{4} \\ 1 \\ \frac{7}{4} \end{bmatrix} \text{ or } \begin{bmatrix} -5 \\ 4 \\ 7 \end{bmatrix}.$$

$\square$

Problem 3. Consider the following linear transformation $T : \mathbb{R}^4 \to \mathbb{R}^3$:

$$T(w, x, y, z) = (2x + 4w, z - y, x + 2w + y - z).$$

(a.) Express the transformation in the form $T(v) = Mv$ for some matrix M (where v is in the form of a column vector $v = [w, x, y, z]^T$).

(b.) Find a basis for the null space.

(c.) What is the dimension of the image space?

Solution. (a.)

$$T\left(\begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} 4 & 2 & 0 & 0 \\ 0 & 0 & -1 & 1 \\ 2 & 1 & 1 & -1 \end{bmatrix} \begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix}$$

(b.)

$$\begin{bmatrix} 4 & 2 & 0 & 0 \\ 0 & 0 & -1 & 1 \\ 2 & 1 & 1 & -1 \end{bmatrix} \Rightarrow \begin{bmatrix} 4 & 2 & 0 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

This gives us the equations

$$\begin{aligned} 4w + 2x &= 0 \\ x &= -2w \\ \text{and} & \\ -y + z &= 0 \\ z &= y. \end{aligned}$$

So a basis for the null space is

$$\begin{bmatrix} 1 \\ -2 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}.$$

(c.) The nullity is 2, the dimension of the domain space is 4, therefor the rank is 2 - which is also the dimension of the image space. (See Thm 6.5 in the text.)

□

Problem 4. Find the eigenvalues and eigenvectors for the following matrix:

$$\begin{bmatrix} 2 & -2 & 3 \\ 0 & 3 & -2 \\ 0 & -1 & 2 \end{bmatrix}$$

Solution. The characteristic equation is

$$\begin{aligned} (2-\lambda)(\lambda^2 - 5\lambda + 4) &= 0 \\ (2-\lambda)(\lambda - 1)(\lambda - 4) &= 0. \end{aligned}$$

So the eigenvalues are $2, 1$ and 4. The corresponding eigenvectors are

$$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} \frac{7}{2} \\ -2 \\ 1 \end{bmatrix}$$

□

Problem 5. Argue that the following matrix A is diagonalizable by finding a matrix P so that $P^{-1}AP$ is a diagonal matrix:

$$A = \begin{bmatrix} 4 & 2 \\ -1 & 1 \end{bmatrix}.$$

Solution. The characteristic equation is

$$\lambda^2 - 5\lambda + 6 \quad = \quad 0$$

So the eigenvalues are 2 and 3. The corresponding eigenvectors are

$$\begin{bmatrix} 1 \\ -1 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

The the matrix P of eigenvectors is

$$P = \begin{bmatrix} 1 & -2 \\ -1 & 1 \end{bmatrix}.$$

We have $\det(P) = -1$. Since the determinant of P is non-zero, the eigenvectors are linearly independent, so the matrix diagonalizes the original matrix. □

Problem 6. Argue that the following matrix A is diagonalizable by finding a matrix P so that $P^{-1}AP$ is a diagonal matrix:

$$A = \begin{bmatrix} 2 & -2 & 1 \\ 0 & 3 & 2 \\ 0 & 0 & 1 \end{bmatrix}.$$

Solution. The characteristic equation is

$$(2-\lambda)(3-\lambda)(1-\lambda) \quad = \quad 0.$$

So the eigenvalues are $2, 3$ and 1. The corresponding eigenvectors are

$$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ -1 \\ 1 \end{bmatrix}.$$

The the matrix P of eigenvectors is

$$P = \begin{bmatrix} 1 & -2 & -3 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}.$$

We have $\det(P) = 1$. Since the determinant of P is non-zero, the eigenvectors are linearly independent, so the matrix diagonalizes the original matrix.

$\square$