MATH2660 Dr. Smith Test 1, May 24, 2024.

Make sure to show all your work. You may not receive full credit if the accompanying work is incomplete or incorrect. If you do scratch work make sure to indicate scratch work - I will not take off points for errors in the scratch work if it is so labeled and will assume that the scratch work is not part of the final answer. You may use your calculators.

Problem 1. Use the Gaussian elimination procedure to solve the following system of equations:

$$\begin{aligned} x+y+z &= -5 \\ 3x+4y+4z &= 1 \\ 3x+6y+5z &= -4. \end{aligned}$$

Solution.

$$\begin{bmatrix} 1 & 1 & 1 & \mid & -5 \\ 3 & 4 & 4 & \mid & 1 \\ 3 & 6 & 5 & \mid & -4 \end{bmatrix}$$

$-3L1 + L2 \to L2$
$-3L1 + L3 \to L3$

$$\begin{bmatrix} 1 & 1 & 1 & \mid & -5 \\ 0 & 1 & 1 & \mid & 16 \\ 0 & 3 & 2 & \mid & 11 \end{bmatrix}$$

$-3L2 + L3 \to L3$

$$\begin{bmatrix} 1 & 1 & 1 & \mid & -5 \\ 0 & 1 & 1 & \mid & 16 \\ 0 & 0 & -1 & \mid & -37 \end{bmatrix}$$

$$\begin{aligned} z &= 37 \\ y+z &= 16 \\ y &= 16-37=-21 \\ x+y+z &= -5 \\ x &= -5+21-37=-21. \end{aligned}$$

□

Problem 2 & 3. Find the inverses of the following matrices:

$$(2.)\ \begin{bmatrix} 3 & -2 \\ 9 & 4 \end{bmatrix}$$

Solution.

$$\begin{bmatrix} 3 & -2 & | & 1 & 0 \\ 9 & 4 & | & 0 & 1 \end{bmatrix}$$

$-3L1 + L2 \to L2$

$$\begin{bmatrix} 3 & -2 & | & 1 & 0 \\ 0 & 10 & | & -3 & 1 \end{bmatrix}$$

$\frac{1}{5}L2 + L1 \to L1$

$$\begin{bmatrix} 3 & 0 & | & \frac{2}{5} & \frac{1}{5} \\ 0 & 10 & | & -3 & 1 \end{bmatrix}$$

$\frac{1}{3}L1 \to L1$ and $\frac{1}{4}L2 \to L2$

$$\begin{bmatrix} 1 & 0 & | & \frac{2}{15} & \frac{1}{15} \\ 0 & 1 & | & -\frac{3}{10} & \frac{1}{10} \end{bmatrix}$$

□

$$(3.)\ \begin{bmatrix} 1 & 1 & 2 \\ 0 & 3 & 0 \\ -2 & -4 & 3 \end{bmatrix}$$

Solution.

$$\begin{bmatrix} 1 & 1 & 2 & | & 1 & 0 & 0 \\ 0 & 3 & 0 & | & 0 & 1 & 0 \\ -2 & -4 & 3 & | & 0 & 0 & 1 \end{bmatrix}$$

$-\frac{1}{3}L2 + L1 \to L1$ and $\frac{4}{3}L2 + L3 \to L3$ then $\frac{1}{3}L2 \to L2$

$$\begin{bmatrix} 1 & 0 & 2 & | & 1 & -\frac{1}{3} & 0 \\ 0 & 1 & 0 & | & 0 & \frac{1}{3} & 0 \\ -2 & 0 & 3 & | & 0 & \frac{4}{3} & 1 \end{bmatrix}$$

$2L1 + L3 \to L3$

$$\begin{bmatrix} 1 & 0 & 2 & | & 1 & -\frac{1}{3} & 0 \\ 0 & 1 & 0 & | & 0 & \frac{1}{3} & 0 \\ 0 & 0 & 7 & | & 2 & \frac{2}{3} & 1 \end{bmatrix}$$

$\frac{1}{7}L3 \to L3$

$$\begin{bmatrix} 1 & 0 & 2 & | & 1 & -\frac{1}{3} & 0 \\ 0 & 1 & 0 & | & 0 & \frac{1}{3} & 0 \\ 0 & 0 & 1 & | & \frac{2}{7} & \frac{2}{21} & \frac{1}{7} \end{bmatrix}$$

$-2L3 + L1 \to L1$

$$\begin{bmatrix} 1 & 0 & 0 & | & \frac{3}{7} & -\frac{11}{21} & -\frac{2}{7} \\ 0 & 1 & 0 & | & 0 & \frac{1}{3} & 0 \\ 0 & 0 & 1 & | & \frac{2}{7} & \frac{2}{21} & \frac{1}{7} \end{bmatrix}$$

□

Problem 4. Find the inverse of the following matrix:

$$\begin{bmatrix} 3 & -1 & 0 \\ 6 & 4 & 2 \\ 0 & 5 & -2 \end{bmatrix}$$

Solution.

$$\begin{bmatrix} 3 & -1 & 0 & | & 1 & 0 & 0 \\ 6 & 4 & 2 & | & 0 & 1 & 0 \\ 0 & 5 & -2 & | & 0 & 0 & 1 \end{bmatrix}$$

$L3 + L2 \to L2$

$$\begin{bmatrix} 3 & -1 & 0 & | & 1 & 0 & 0 \\ 0 & 11 & 0 & | & -2 & 1 & 1 \\ 0 & 5 & -2 & | & 0 & 0 & 1 \end{bmatrix}$$

$\frac{1}{11}L2 \to L2$

$$\begin{bmatrix} 3 & -1 & 0 & | & 1 & 0 & 0 \\ 0 & 1 & 0 & | & -\frac{2}{11} & \frac{1}{11} & \frac{1}{11} \\ 0 & 5 & -2 & | & 0 & 0 & 1 \end{bmatrix}$$

$L2 + L1 \to L1$ and $-5L2 + L3 \to L3$

$$\left[\begin{array}{ccc|ccc} 3 & 0 & 0 & \frac{9}{11} & \frac{1}{11} & \frac{1}{11} \\ 0 & 1 & 0 & -\frac{2}{11} & \frac{1}{11} & \frac{1}{11} \\ 0 & 0 & -2 & \frac{10}{11} & -\frac{5}{11} & \frac{6}{11} \end{array}\right]$$

$\frac{1}{3}L1 \to L1$ and $-\frac{1}{2}L3 \to L3$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{9}{33} & \frac{1}{33} & \frac{1}{33} \\ 0 & 1 & 0 & -\frac{2}{11} & \frac{1}{11} & \frac{1}{11} \\ 0 & 0 & 1 & -\frac{10}{22} & \frac{5}{22} & -\frac{6}{22} \end{array}\right]$$

It's easier to check if you find a common denominator:

$$\text{Inverse} = \begin{bmatrix} \frac{18}{66} & \frac{2}{66} & \frac{2}{66} \\ -\frac{12}{66} & \frac{6}{66} & \frac{6}{66} \\ -\frac{30}{66} & \frac{15}{66} & -\frac{18}{66} \end{bmatrix}$$

□

Problem 4 & 5. Find the determinant of the following matrices:

$$(4.) \begin{bmatrix} 3 & -1 & 2 \\ 2 & 4 & -2 \\ -2 & 3 & -3 \end{bmatrix}$$

Solution. Expanding by the cofactors of row 1 we have:

$$3\begin{vmatrix} 4 & -2 \\ 3 & -3 \end{vmatrix} + \begin{vmatrix} 2 & -2 \\ -2 & -3 \end{vmatrix} + 2\begin{vmatrix} 2 & 4 \\ -2 & 3 \end{vmatrix}$$

$$= 3(-12 + 6) + (-6 - 4) + 2(6 + 8) = -18 - 10 + 28 = 0.$$

□

$$(5.) \begin{bmatrix} 3 & 1 & -4 & 2 \\ 4 & 0 & 1 & -1 \\ 0 & 0 & -2 & 0 \\ 0 & 2 & -3 & 0 \end{bmatrix}$$

Solution. Expanding by the cofactors of row 3 we have :

$$-2\begin{vmatrix} 3 & 1 & 2 \\ 4 & 0 & -1 \\ 0 & 2 & 0 \end{vmatrix} = -2\cdot(-2)\begin{vmatrix} 3 & 2 \\ 4 & -1 \end{vmatrix} = 4(-3-8) = -44$$

□

Problem 6. Prove that if A is a 2×2 matrix and B is obtained from A by replacing the second row with k times the first row plus the second row, then they have the same determinants.

Solution. Consider the 2×2 matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and constant k. Then:

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc.$$

and

$$\begin{vmatrix} a & b \\ ka+c & kb+d \end{vmatrix} = a(kb+d) - b(ka+c) = akb + ad - bka - bc = ad - bc.$$

□

Problem 7 a. Argue that the following two matrices are inverses of each other

$$\begin{bmatrix} 1 & -3 & 3 \\ 1 & 2 & -3 \\ -1 & 1 & 2 \end{bmatrix} \quad \begin{bmatrix} 1 & 3 & 3 \\ 1 & 5 & 6 \\ 1 & 4 & 5 \end{bmatrix}$$

Solution. They are not inverses of each other; if you multiply to two you get

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 10 & 13 \end{bmatrix}$$

□

Problem 7 b. Solve the following systems of equations:

$$\begin{bmatrix} 1 & 3 & 3 \\ 1 & 5 & 6 \\ 1 & 4 & 5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ -2 \\ 5 \end{bmatrix}$$

Solution.

$$\begin{bmatrix} 1 & 3 & 3 & | & 3 \\ 1 & 5 & 6 & | & -2 \\ 1 & 4 & 5 & | & 5 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 3 & 3 & | & 3 \\ 0 & 2 & 3 & | & -5 \\ 0 & 1 & 2 & | & 2 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 3 & 3 & | & 3 \\ 0 & 2 & 3 & | & -5 \\ 0 & 1 & 2 & | & 2 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 3 & 3 & | & 3 \\ 0 & 2 & 3 & | & -5 \\ 0 & 0 & \frac{1}{2} & | & \frac{9}{2} \end{bmatrix}$$

$$\begin{aligned} \frac{1}{2}z &= \frac{9}{2} \\ z &= 9 \\ 2y + 3z &= -5 \\ 2y &= -5 - 27 \\ y &= -16 \\ x + 3y + 3z &= 3 \\ x &= 24. \end{aligned}$$

□

Problem 8 a. It is easy to see that the following system of equations has a solution: namely $x = y = z = 0$, does it have more than one and if so what are they?

$$\begin{aligned} x + 2y - 4z &= 0 \\ 2x + 2y + 3z &= 0 \\ 4x + 6y - 5z &= 0. \end{aligned}$$

Solution.

$$\begin{bmatrix} 1 & 2 & -4 & | & 0 \\ 2 & 2 & 3 & | & 0 \\ 4 & 6 & -5 & | & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 2 & -4 & | & 0 \\ 0 & -2 & 11 & | & 0 \\ 0 & -2 & 11 & | & 0 \end{bmatrix}$$

$$\Rightarrow \left[\begin{array}{ccc|c} 1 & 2 & -4 & 0 \\ 0 & -2 & 11 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

So z can be anything:

$$\begin{aligned} z &= t \\ -2y + 11z &= 0 \\ y &= \frac{11}{2}t \\ x + 2y - 4z &= 0 \\ x &= 4z - 2y = 4t - 11t = -7t \end{aligned}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -7t \\ \frac{11}{2}t \\ t \end{bmatrix}.$$

□

Problem 8 b. Does the following system of equations have a solution? If yes, what is it; if no, why not. [Note the small change.]

$$\begin{aligned} x + 2y - 4z &= 0 \\ 2x + 2y + 3z &= 0 \\ 4x + 6y - 5z &= 1. \end{aligned}$$

Solution.

$$\Rightarrow \left[\begin{array}{ccc|c} 1 & 2 & -4 & 0 \\ 0 & -2 & 11 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right]$$

If there is a solution, then the last lime yields the equation $0 = 1$, which is a contradiction. □

Problem 9. Find all the values of k for which the following matrix has a determinant equal to 0:

$$\begin{bmatrix} 3-k & 2 \\ 3 & 4-k \end{bmatrix}.$$

Solution.

$$\begin{aligned} (3-k)(4-k)-6 &= 0 \\ 12-4h-3k+k^2-6 &= 0 \\ k^2-7k+6 &= 0 \\ (k-6)(k-1) &= 0 \\ k=6 \quad \text{or} \quad k &= 1. \end{aligned}$$

□

Problem 10. For each of the values of k obtained for problem 9, find the general solution of the system of equations (in terms of a parameter):

$$\begin{bmatrix} 3-k & 2 \\ 3 & 4-k \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Solution. $k = 6$ gives,

$$\begin{bmatrix} -3 & 2 \\ 3 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

$$\begin{bmatrix} -3 & 2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

So, letting y equal our parameter t we have.

$$\begin{aligned} -3x+2y &= 0 \\ \begin{bmatrix} x \\ y \end{bmatrix} &= \begin{bmatrix} \frac{2}{3}t \\ t \end{bmatrix} \end{aligned}$$

$k = 1$ gives,

$$\begin{bmatrix} 2 & 2 \\ 3 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

$$\begin{bmatrix} 2 & 2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

So, letting y equal our parameter t we have.

$$\begin{aligned} 2x+2y &= 0 \\ \begin{bmatrix} x \\ y \end{bmatrix} &= \begin{bmatrix} -t \\ t \end{bmatrix} \end{aligned}$$

□