MATH2660 Dr. Smith Test 2, June 7, 2024.

Please show all your work; you may not receive full credit if the accompanying work is incomplete or incorrect. If you do scratch work make sure to indicate scratch work - I will not take off points for errors in the scratch work if it is so labeled and will assume that the scratch work is not part of the final answer. You may use non-programmable calculators.

Assume, unless otherwise indicated, that the standard inner product is used when inner products are discussed.

Problem 1. For the following row reduced matrix, find a basis for the row space and a basis for the null space:

$$\begin{bmatrix} 1 & 2 & -3 & 2 \\ 0 & 0 & -2 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Solution. Basis for the row space is { (1, 2, -3, 2), (0, 0, -2, 4) }.

The null space are the vectors (x, y, z, w) satisfying:

$$\begin{aligned} x + 2y - 3z + 2w &= 0 \\ -2z + 4w &= 0. \end{aligned}$$

Letting $y = 1, w = 0$ and $y = 0, w = 1$ yields the vectors $(-2, 1, 0, 0), (4, 0, 2, 1)$

□

Problem 2. Consider the following set of vectors from $\mathbb{R}^4$:

$$(1, 1, 0, -2), (2, -2, 3, 1), (2, 1, -1, -3).$$

a. Is the set an orthogonal set? Why or why not.

Solution. No since the inner product of any one pair is not zero. (This is sufficient).

$$\begin{aligned} (2, -2, 3, 1) \cdot (2, 1, -1, -3) &= 4 - 2 - 3 - 3 = -4 \neq 0 \\ (1, 1, 0, -2) \cdot (2, -2, 3, 1) &= 2 - 2 - 2 = -2 \neq 0 \\ (1, 1, 0, -2) \cdot (2, 1, -1, -3) &= 2 + 1 + 0 + 6 = 9 \neq 0. \end{aligned}$$

□

b. Does the set span $\mathbb{R}^4$? Why or why not.

Solution. No because $\mathbb{R}^4$ has dimension 4 and there are only 3 vectors. □

Problem 3. Consider the following attempts to construct inner products on the linear space of continuous functions defined on the interval $[-1, 1]$.

$$\begin{aligned} \text{(a.)} \quad < f, g >_a &= \int_{-1}^{1} f(x)g(x)dx + 4f(1) \cdot g(1) \\ \text{(b.)} \quad < f, g >_b &= \int_{0}^{1} f(x)g(x)dx. \end{aligned}$$

(a.) Show that condition(a) is an inner product but condition (b) is not.

Solution. You just have to state the conditions of the inner product and show that they work foor part (a). The condition that (b) doesn't satisfy is the one that requires $< f, f >> 0$ unless $f = 0$. If f is any continuous function that is non zero any where on $[-1, 0]$ and zero on $[0, 1]$ is a non-zero function so that $< f, f >_b$ is zero. □

(b.) Then use condition (a) to calculate $< f, f >_a$ for $f(x) = 1 + x^2$.

Solution.

$$\begin{aligned} < f, f >_a &= \int_{-1}^{1} (1 + x^2)^2 dx + 4 \cdot 2^2 \\ &= \int_{-1}^{1} 1 + 2x^2 + x^4 dx + 16 \\ &= x + \frac{2}{3}x^3 + \frac{1}{5}x^5|_{-1}^{1} + 16 \\ &= 2 + \frac{4}{3} + \frac{2}{5} + 16 \\ &= 19 + \frac{11}{15}. \end{aligned}$$

□

(c.) Use the inner product $< f, g >_a$ to find the vector of unit norm that is in the same direction as $f(x) = 1 + x^2$.

Solution.

$$\frac{v}{||v||} = \frac{1}{\sqrt{19\frac{11}{15}}}(1+x^2).$$

□

Problem 4. For the vectors $v = (1, 3, -2), u = (1, -1, 2)$ calculate:
(a.) the orthogonal projection of u onto v,

Solution.

$$\begin{aligned} \text{proj}_v u &= \frac{< u, v >}{< v, v >} v \\ &= \frac{1-3-4}{1+9+4} v \\ &= \frac{-6}{14} v. \end{aligned}$$

□

(b.) the cosine of the angle between u and v.

Solution.

$$\begin{aligned} \cos\theta &= \frac{< u, v >}{||v|| \cdot ||u||} \\ &= \frac{-6}{\sqrt{14 \cdot 6}} \\ &= \frac{-3}{\sqrt{21}}. \end{aligned}$$

□

Problem 5. Are the following elements of the linear space of polynomials linearly independent?

$$\begin{aligned} f_1(t) &= 1 + t^2 \\ f_2(t) &= t + t^2 \\ f_3(t) &= 1 + t. \end{aligned}$$

Solution. Consider

$$M = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

Since the determinant of $M = -2 \neq 0$ the vectors are linearly independent. □

Problem 6. Find a set of orthogonal vector that span the space generated by the basis $\{(1,1,0,0),(0,0,1,1),(2,1,1,1)\}$.

Solution. The first two vectors are already orthogonal, so $w_1 = (1,1,0,0), w_2 = (0,0,1,1)$. So we only need to compute w_3.

$$\begin{aligned} w_3 &= v_3 - \frac{< v_3, w_1 >}{< w_1, w_1 >} w_1 - \frac{< v_3, w_2 >}{< w_2, w_2 >} w_2 \\ &= (2,1,1,1) - \frac{3}{2}(1,1,0,0) - \frac{2}{2}(0,0,1,1 \\ &= (\frac{1}{2}, -\frac{1}{2}, 0, 0). \end{aligned}$$

The orthogonal basis is $B = \{(1,1,0,0),(0,0,1,1),(\frac{1}{2},-\frac{1}{2},0,0)\}$ □

Problem 7. Consider the set: $B = \{\frac{1}{3}(1,0,-2,2), \frac{1}{3}(0,1,2,2), \frac{1}{3}(2,2,0,-1)\}$.

a.) Show that this set of vectors is orthonormal

Solution. It's straight forward to check that the inner product of each pair is 0. And each has norm $\sqrt{9\frac{1}{9}} = 1$. □

b.) Express the vector $(8,7,-2,3)$ as a linear combination of these orthonormal vectors.

Solution. Let v denote the vector $(8,7,-2,3)$. Then for $B = \{w_1, w_2, w_3\}$.

$$\begin{aligned} (8,7,-2,3) &= < v, w_1 > w_1 + < v, w_2 > w_2 + < v, w_3 > w_3 \\ &= \frac{1}{3}(8+4+6)w_1 + \frac{1}{3}(7-4+6)w_2 + \frac{1}{3}(16+14-3)w_3 \\ &= 6w_1 + 3w_2 + 9w_3 \\ &= 6\frac{1}{3}(1,0,-2,2) + 3\frac{1}{3}(0,1,2,2) + 9\frac{1}{3}(2,2,0,-1) \end{aligned}$$

To check we multiply out and obtain:

$$\begin{aligned} &= 2(1,0,-2,2)+1(0,1,2,2)+3(2,2,0,-1) \\ &= (8,7,-2,3). \end{aligned}$$

□

Problem 8. Suppose that the arbitrary 2×2 matrix M has rank 1.:

$$M = \begin{bmatrix} a & b \\ c & d \end{bmatrix},$$

Find a basis in terms of the quantities in the matrix.

Then:

(a.) Show that one row is a multiple of the other.

Solution. Since it has rank one either row is a basis and the other row is a multiple of the basis row. So we can assume that we have:

$$M = \begin{bmatrix} a & b \\ ka & kb \end{bmatrix},$$

□

(b.) Show that the determinant is 0.

Solution.

$$\det(m) = akb - bka = 0.$$

□

(c.) Show that the null space has dimension 1.

(d.) Calculate the basis for the null space in terms of a and b.

Solution. Solution of c and d: The null space is the set of vectors (x, y) so that $ax + by = 0$ and $kax + kby = 0$ but these are equivalent equations. So

$$\begin{aligned} ax + by &= 0 \\ x &= -\frac{b}{a}y. \end{aligned}$$

And the solutions are multiples of the vector $\begin{bmatrix} -\frac{b}{a} \\ 1 \end{bmatrix}$ □

Problem 9. For the following row reduced matrix:

$$\begin{bmatrix} 2 & 1 & -2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

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(a.) Find a basis for the row space and a basis for the null space.

Solution. Basis of the row space is $B = \{(2,1,-2),(0,1,-1)\}$ For the null space:

$$\begin{aligned} 2x + y - 2z &= 0 \\ y - z &= 0. \end{aligned}$$

Let $z = 1$ to obtain a basis. So the single vector $(\frac{1}{2}, 1, 1)$ is a basis. □

(b.) Is a vector in the null space orthogonal to the basis vectors of the row space? Why or why not.

Solution. Yes because:

$$\begin{aligned} (\frac{1}{2}, 1, 1) \cdot (2, 1, -2) &= 1 + 1 - 2 = 0 \\ (\frac{1}{2}, 1, 1) \cdot (0, 1, -1) &= 1 - 1 = 0. \\ y - z &= 0. \end{aligned}$$

□

Problem 10. Prove that if the vector $\overrightarrow{v}$ is orthogonal to the vectors $\overrightarrow{u_1}$ and $\overrightarrow{u_2}$, then it is orthogonal to every vector of the space spanned by the vectors $\overrightarrow{u_1}$ and $\overrightarrow{u_2}$.

Solution.

$$\begin{aligned} < \overrightarrow{v}, a\overrightarrow{u_1} + b\overrightarrow{u_2} > &= a < \overrightarrow{v}, \overrightarrow{u_1} > +b < \overrightarrow{v}, \overrightarrow{u_2} > \\ &= a0 + b0 \\ &= 0. \end{aligned}$$

□