MATH2660 Dr. Smith.
Test 3 Key.

Please do all your work on separate paper (which you should provide) indicate by number the problem you are working on and specifically highlight your answer (with lines, boxes, labels or such) it is not necessary to rewrite the problem. Note that you may not receive full credit if the accompanying work is incomplete or incorrect. Assume that I will not read anything on the test sheets.

If you do scratch work make sure to indicate scratch work - I will not take off points for errors in the scratch work if it is so labeled and will assume that the scratch work is not part of the final answer.

You may use non-programmable calculators.

Assume, unless otherwise indicated, that the standard inner product is used when inner products are discussed.

Problem 1. Consider the linear transformation defined by the following matrix equation:

$$T\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ 1 & -3 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

a.) Find a basis for the kernel.
b.) Find a basis for the range.

Solution. Part a. The kernel is the set of vectors that satisfies the three equations: $2x + 2y = 0$, $x - 3y = 0$ and $3x + y = 0$. The only vector v that satisfies all three equations (in fact that satisfy any pair of them) is $v = (0, 0)$.

Part b. The range is spanned by the column vectors, so an easy to see basis is the set of vectors $\{(2, 1, 3), (2, -3, 1)\}$. □

Problem 2. Consider the linear transformation defined by the following matrix equation:

$$T\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 & 2 & -1 \\ 0 & 1 & -3 \\ 0 & 2 & -6 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

a.) Find a basis for the kernel.
b.) Find a basis for the range.

Solution. Part a. Since the third row is a multiple of the second row (you would see this if you used Gaussian elimination), so the kernel is the set of vectors that satisfies the three equations: $2x + 2y - z = 0$, and $y - 3z = 0$. So: $y = 3z$ and $2x = z - 2y = z - 6z = -5z$. So the kernel is

$$\begin{bmatrix} -\frac{5}{2} \\ 3 \\ 1 \end{bmatrix} t.$$

Part b. A basis can obtained by using Gaussian elimination on the transpose matrix, e.g. :

$$\begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}.$$

□

Problem 3. Consider the vector space of twice differential functions
a.) Show that T defined by $T(f) = 4f + f''$ is a linear transformation.
b.) Show that the functions $g(x) = \cos(2x)$ and $h(x) = \sin(2x)$ are in the kernel of T.

Solution. Part a.: For functions f and g,

$$\begin{aligned} T(f+g) &= 4(f+g) + (f+g)'' \\ &= 4f + 4g + f'' + g'' \\ &= 4f + f'' + 4g + g'' \\ &= T(f) + T(g). \end{aligned}$$

and for a constant scalar c,

$$\begin{aligned} T(a) &= 4(cf) + (cf)'' \\ &= c4f + cf'' \\ &= c(4f + f'') \\ &= cT(f). \end{aligned}$$

Part b.

$$\begin{aligned}
T(g) &= 4\cos(2x) + (\cos(2x))'' \\
&= 4\cos(2x) + (-2\sin(2x))' \\
&= 4\cos(2x) + (-4\cos(2x)) \\
&= 0 \\
T(h) &= 4\sin(2x) + (\sin(2x)))'' \\
&= 4\sin(2x) + (2\cos(2x))' \\
&= 4\sin(2x)) + (-4\sin(2x)) \\
&= 0.
\end{aligned}$$

□

Problem 4. Find the Eigenvalues and corresponding Eigenvectors of the following matrix

$$\begin{bmatrix} -1 & 4 \\ 1 & 2 \end{bmatrix}$$

Solution. Eigenvalues:

$$\begin{aligned}
(-1-\lambda)(2-\lambda) - 4 &= \lambda^2 - \lambda - 2 - 4 \\
&= \lambda^2 - \lambda - 6 \\
&= (\lambda - 3)(\lambda + 2) = 0.
\end{aligned}$$

So the Eigenvalues are $\lambda = 3$ and $\lambda = -2$.

$\lambda = -2$ yields:

$$\begin{bmatrix} -1-\lambda & 4 \\ 1 & 2-\lambda \end{bmatrix} = \begin{bmatrix} 1 & 4 \\ 1 & 4 \end{bmatrix}.$$

So the Eigenvector satisfies $x + 4y = 0$. Then the Eigenspace is

$$\begin{bmatrix} -4 \\ 1 \end{bmatrix} t$$

for $t \in \mathbb{R}$.

$\lambda = 3$ yields:

$$\begin{bmatrix} -1-\lambda & 4 \\ 1 & 2-\lambda \end{bmatrix} = \begin{bmatrix} -4 & 4 \\ 1 & -1 \end{bmatrix}.$$

So the Eigenvector satisfies $x - y = 0$. Then the Eigenspace is

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix} t$$

for $t \in \mathbb{R}$.

$\square$

Problem 5.

a.) Find the Eigenvalues and corresponding Eigenvectors of the following matrix

$$\begin{bmatrix} -1 & 2 \\ 2 & 2 \end{bmatrix}$$

b.) Show that the Eigenvectors corresponding to different Eigenvalues are orthogonal.

Solution. Part a. The characteristic equation is:

$$\begin{aligned} (-1-\lambda)(2-\lambda) - 4 &= \lambda^2 - \lambda - 6 = 0 \\ (\lambda - 3)(\lambda + 2) &= 0 \\ \lambda &= 3, -2. \end{aligned}$$

For $\lambda = -2$ we have

$$\begin{bmatrix} -1-\lambda & 2 \\ 2 & 2-\lambda \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 0$$

$$x + 2y = 0.$$

So the Eigenvector is $v_1 = \begin{bmatrix} -2 \\ 1 \end{bmatrix} t$.

For $\lambda = 3$ we have

$$\begin{bmatrix} -1-\lambda & 2 \\ 2 & 2-\lambda \end{bmatrix} \Rightarrow \begin{bmatrix} -4 & 2 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 0$$

$$2x - y = 0.$$

So the Eigenvector is $v_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix} t.$ □

Part b.

$$< v_1, v_2 > = < (-2, 1), (1, 2) >= 0.$$

Problem 6.

a.) Find the Eigenvalues and corresponding Eigenvectors of the following matrix

$$\begin{bmatrix} 5 & 0 & 3 \\ 0 & 2 & 0 \\ 3 & 0 & 5 \end{bmatrix}$$

b.) Show that the Eigenvectors corresponding to different Eigenvalues are orthogonal.

Solution.

$$\begin{vmatrix} 5-\lambda & 0 & 3 \\ 0 & 2-\lambda & 0 \\ 3 & 0 & 5-\lambda \end{vmatrix} \to (2-\lambda) \begin{vmatrix} 5-\lambda & 3 \\ 3 & 5-\lambda \end{vmatrix}.$$

This gives us the characteristic equation:

$$\begin{aligned} (2-\lambda)((5-\lambda)^2 - 9) &= (2-\lambda)((5-\lambda)^2 - 9) \\ &= (2-\lambda)(\lambda^2 - 10\lambda - 16) \\ &= (2-\lambda)(\lambda - 2)(\lambda - 8) = 0 \\ \lambda &= 2, 8. \end{aligned}$$

For $\lambda = 2$ we have:

$$\begin{bmatrix} 5-\lambda & 0 & 3 \\ 0 & 2-\lambda & 0 \\ 3 & 0 & 5-\lambda \end{bmatrix} \to \begin{bmatrix} 3 & 0 & 3 \\ 0 & 0 & 0 \\ 3 & 0 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0.$$

This give us the equations

$$\begin{aligned} 3x + 3z &= 0 \\ 0x + 0y + 0z &= 0. \end{aligned}$$

For which the corresponding Eigenvectors are

$$v_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} t, v_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} s.$$

For $\lambda = 8$ we have

$$\begin{bmatrix} 5-\lambda & 0 & 3 \\ 0 & 2-\lambda & 0 \\ 3 & 0 & 5-\lambda \end{bmatrix} \to \begin{bmatrix} -3 & 0 & 3 \\ 0 & -6 & 0 \\ 3 & 0 & -3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0.$$

This give us the equations

$$\begin{aligned} -3x + 3z &= 0 \\ -6y &= 0. \end{aligned}$$

For which the corresponding Eigenvector is

$$v_3 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} t$$

Part b.: You just need to verify that $< v_1, v_3 >= 0$ and $< v_2, v_3 >= 0$.

□

Problem 7.

a.) Find the Eigenvalues and corresponding Eigenvectors of the following matrix

$$\begin{bmatrix} 5 & 0 & 3 \\ 0 & 1 & 0 \\ 3 & 0 & 5 \end{bmatrix}$$

b.) Show that the Eigenvectors corresponding to different Eigenvalues are orthogonal.

Solution.

$$\begin{vmatrix} 5-\lambda & 0 & 3 \\ 0 & 1-\lambda & 0 \\ 3 & 0 & 5-\lambda \end{vmatrix} \to (1-\lambda) \begin{vmatrix} 5-\lambda & 3 \\ 3 & 5-\lambda \end{vmatrix}.$$

This gives us the characteristic equation:

$$\begin{aligned}(1-\lambda)((5-\lambda)^2-9) &= (1-\lambda)((5-\lambda)^2-9) \\ &= (1-\lambda)(\lambda^2-10\lambda-16) \\ &= (1-\lambda)(\lambda-2)(\lambda-8)=0 \\ \lambda &= 1,2,8.\end{aligned}$$

For $\lambda = 1$ we have:

$$\begin{bmatrix} 5-\lambda & 0 & 3 \\ 0 & 1-\lambda & 0 \\ 3 & 0 & 5-\lambda \end{bmatrix} \rightarrow \begin{bmatrix} 4 & 0 & 3 \\ 0 & 0 & 0 \\ 3 & 0 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0.$$

This give us the equations

$$\begin{aligned} 4x+3z &= 0 \\ 0x+0y+0z &= 0 \\ x+4z &= 0 \end{aligned}$$

For which the corresponding Eigenvector is

$$v_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} t.$$

For $\lambda = 2$ we have:

$$\begin{bmatrix} 5-\lambda & 0 & 3 \\ 0 & 1-\lambda & 0 \\ 3 & 0 & 5-\lambda \end{bmatrix} \rightarrow \begin{bmatrix} 3 & 0 & 3 \\ 0 & 0 & 0 \\ 3 & 0 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0.$$

This give us the equations

$$\begin{aligned} 3x+3z &= 0 \\ 0x+0y+0z &= 0. \end{aligned}$$

For which the corresponding Eigenvector is

$$v_2 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} t.$$

For $\lambda = 8$ we have

$$\begin{bmatrix} 5-\lambda & 0 & 3 \\ 0 & 1-\lambda & 0 \\ 3 & 0 & 5-\lambda \end{bmatrix} \to \begin{bmatrix} -3 & 0 & 3 \\ 0 & -7 & 0 \\ 3 & 0 & -3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0.$$

This give us the equations

$$\begin{aligned} -3x + 3z &= 0 \\ -7y &= 0. \end{aligned}$$

For which the corresponding Eigenvector is

$$v_3 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} t$$

□

Problem 8. Verify the Cayley-Hamilton Theorem by showing that the following matrix satisfies its characteristic equation

$$\begin{bmatrix} 2 & 3 \\ 5 & 4 \end{bmatrix}$$

Solution.

$$\begin{bmatrix} 2-\lambda & 3 \\ 5 & 4-\lambda \end{bmatrix} \Rightarrow \lambda^2 - 6\lambda - 7 = 0.$$

We need to show that the matrix A above satisfies $A^2 - 6A - 7I = 0$.

$$A^2 = \begin{bmatrix} 2 & 3 \\ 5 & 4 \end{bmatrix} \begin{bmatrix} 2 & 3 \\ 5 & 4 \end{bmatrix} = \begin{bmatrix} 19 & 18 \\ 30 & 31 \end{bmatrix}$$

$$-6A = \begin{bmatrix} -12 & -18 \\ -30 & -24 \end{bmatrix}$$

$$-7I = \begin{bmatrix} -7 & 0 \\ 0 & -7 \end{bmatrix}$$

Adding the right column gives 0 as required.

□

Problem 9. Consider the set: $B = \{\frac{1}{2}(1,1,1,1), \frac{1}{2}(-1,1,1,-1), \frac{1}{2}(-1,-1,1,1),.$

a.) Show that this set of vectors is orthonormal .

b.) Express the vector $(9,6,-1,2)$ as a linear combination of these orthonormal vectors.

c.) Does this set of vectors span $\mathbb{R}^4$? Why or why not?

Solution. Part a. is an easy check.

Part b. Let B_1, B_2, B_3 denote the basis vectors and let $v = (9,6,-1,2)$. Since the vectors are orthonormal then the solution is

$$\begin{aligned} v &= < v, B_1 > B_1 + < v, B_2 > B_2 + < v_3, B_3 > B_3 \\ &= 8B_1 - 3B_2 - 7B_3. \end{aligned}$$

Part c. No since the span of three independent vector has dimension 3 and $\mathbb{R}^4$ has dimension 4.

□

Problem 10. Suppose that a is positive. For the following matrix, argue that the pair of Eigenspaces is independent of the value of a.

$$\begin{bmatrix} 2 & a \\ a & 2 \end{bmatrix}$$

[In other words, show that different a values still give you the same Eigenspaces.]

Solution.

$$\begin{bmatrix} 2-\lambda & a \\ a & 2\lambda \end{bmatrix} \Rightarrow (2-\lambda)^2 - a^2$$

$$\begin{aligned} (2-\lambda)^2 - a^2 &= 0 \\ (2-\lambda)^2 &= a^2 \\ 2-\lambda &= \pm a \\ \lambda &= 2+a, 2-a. \end{aligned}$$

For $\lambda = 2 + a$ we have

$$\begin{bmatrix} 2-\lambda & a \\ a & 2-\lambda \end{bmatrix} \Rightarrow \begin{bmatrix} -a & a \\ a & -a \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 0$$

$$x - y = 0.$$

So an Eigenvector is $v_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and the Eigenspace is $\{v_1 t | t \in \mathbb{R}\}$.

For $\lambda = 2 - a$ we have

$$\begin{bmatrix} 2-\lambda & a \\ a & 2-\lambda \end{bmatrix} \Rightarrow \begin{bmatrix} a & a \\ a & a \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 0$$

$$x + y = 0.$$

So an Eigenvector is $v_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ and the Eigenspace is $\{v_2 t | t \in \mathbb{R}\}$.

Therefore the Eigenspaces are independent of a.

$\square$