Babylonian method for finding the square root.

The Babylonians used the following formula estimate to calculate the square root of a number S:

$$\sqrt{S} \quad \approx \quad \frac{\frac{S}{x} + x}{2}.$$

It's actually an interative formula - once a guess for x is made "close enough" the formula produces a better guess. Thus, with an appropriate first guess, repeated iterations yelds a sequence that converges to the root. It is also known as Heron's formula.

The theory for why it works is given by the following theorem.

Theorem: Suppose that $f : \mathbb{R} \to \mathbb{R}$ is a function with the fixed point a, in other words $f(a) = a$. Suppose further that $|f'(a)| < 1$. Then if f is sufficiently differentiable (e.g. continuous derivative at a) then an interval containing a so that if a_0 is chosen in that interval then the following iterative process produces a sequence of points with sequential limit a:

$$a_{n+1}S \quad = \quad f(a_n).$$

Application of the theorem.

Let

$$f(x) \quad = \quad \frac{\frac{S}{x} + x}{2}.$$

first we show that for $a = \sqrt{S}$ we have $f(a) = a$

$$\begin{aligned} f(x) &= \frac{\frac{S}{x} + x}{2} \\ f(\sqrt{S}) &= \frac{\frac{S}{\sqrt{S}} + \sqrt{S}}{2} \\ &= \frac{\sqrt{S} + \sqrt{S}}{2} = \sqrt{S}. \end{aligned}$$

Next we calculate the derivative of f at $x = \sqrt{S}$.

$$
\begin{aligned}
f'(x) &= \frac{\frac{S}{x} + x}{2} \\
&= \frac{-\frac{S}{x^2} + 1}{2} \\
f'(\sqrt{S}) &= \frac{-\frac{S}{(\sqrt{S})^2} + 1}{2} \\
&= 0.
\end{aligned}
$$

Thus by our theorem iterating this formula would yield $\sqrt{S}$ for a "cloase enough" guess. It turns out that the interval in which to pick your first guess is fairly large and any resonable first guess will yield a sequence converging yo $\sqrt{S}$. In fact $a_0 = S$ will work.

How can we create such an iterative process.

It turns out that this formula is equivalent to using Newton's method to find the roots of an equation applied to the equation $x^2 - S = 0$. Newton's method for approximating the root starts with an initial guess x_0 uses the following iterative process on the function $f(x) = x^2 - S$:

$$
x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}.
$$

So in this case $f'(x) = 2x$ and Newton's method yields:

$$
\begin{aligned}
x_{n+1} &= x_n - \frac{x_n^2 - S}{2x_n} \\
&= x_n - \frac{x_n}{2} + \frac{S}{2x_n} \\
&- \frac{1}{2}\Big(x_n + \frac{S}{x_n}\Big).
\end{aligned}
$$

Which is the Babylonian formula.

Exercise. Find a "Babylonian" formula for the cube root of a number and prove that it satisfies the Theorem stated above.