Binomial Series and Curvature.

First the Modern derivation of a version of the binomial theorem. Let $f(x) = (a+ax)^{\frac{m}{n}}$ then in preparation for using the Maclaurin series expansion:

$$\begin{aligned}
f(x) &= (a+ax)^{\frac{m}{n}} \\
f'(x) &= \frac{m}{n}\Big(a+ax\Big)^{\frac{m}{n}-1} a \\
f''(x) &= \frac{m}{n}\Big(\frac{m}{n}-1\Big)\Big(a+ax\Big)^{\frac{m}{n}-2} a^2 \\
f'''(x) &= \frac{m}{n}\Big(\frac{m}{n}-1\Big)\Big(\frac{m}{n}-2\Big)\Big(a+ax\Big)^{\frac{m}{n}-3} a^3 \\
&\vdots
\end{aligned}$$

from which we have

$$\begin{aligned}
f(0) &= a^{\frac{m}{n}} \\
f'(0) &= \frac{m}{n} a^{\frac{m}{n}-1} a = \frac{m}{n} a^{\frac{m}{n}} \\
f''(0) &= \frac{m}{n}\Big(\frac{m}{n}-1\Big) a^{\frac{m}{n}-2} a^2 = \Big(\frac{m(m-n)}{n^2}\Big) a^{\frac{m}{n}} \\
f'''(0) &= \frac{m}{n}\Big(\frac{m}{n}-1\Big)\Big(\frac{m}{n}-2\Big) a^{\frac{m}{n}-3} a^3 = \Big(\frac{m(m-n)(m-2n)}{n^3}\Big) a^{\frac{m}{n}} \\
&\vdots
\end{aligned}$$

From Maclaurin's theorem:

$$\begin{aligned}
f(x) &= f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \dots \\
(a+ax)^{\frac{m}{n}} &= a^{\frac{m}{n}} + \frac{m}{n} a^{\frac{m}{n}} x + \frac{m(m-n)}{2!n^2} a^{\frac{m}{n}} x^2 + \frac{m(m-n)(m-2n)}{3!n^3} a^{\frac{m}{n}} x^3 + \dots
\end{aligned}$$

For a rational number $\frac{a}{b}$, Newton's formulation was (see page 400 of text):

$$(P+PQ)^{\frac{m}{n}} = \underbrace{P^{\frac{m}{n}}}_{A} + \underbrace{\frac{m}{n}AQ}_{B} + \underbrace{\frac{m-n}{2n}BQ}_{C} + \underbrace{\frac{m-2n}{3n}CQ}_{D} + \dots$$

with

$$\begin{aligned}
A &= P^{\frac{m}{n}} \\
B &= \frac{m}{n} AQ \\
C &= \frac{m-n}{2n} BQ \\
D &= \frac{m-2n}{3n} CQ \\
&\vdots
\end{aligned}$$

expanding:

$$\begin{aligned}
A &= P^{\frac{m}{n}} \\
B &= \frac{m}{n} P^{\frac{m}{n}} Q \\
C &= \frac{m(m-n)}{2n^2} P^{\frac{m}{n}} Q^2 \\
D &= \frac{m(m-n)(m-2n)}{3!n^3} P^{\frac{m}{n}} Q^3 \\
&\vdots
\end{aligned}$$

And this matches our formulation. For the special case of

$$\frac{1}{1+x^2} = (1+x^2)^{-1}$$

replacing $P = a$ with 1 and $Q = x$ with x^2 and $m = -1$, $n = 1$ we get

$$\frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + \dots \tag{1}$$

on the other hand replacing a with x^2 and $ax = 1$ so x needs to be replaced with x^{-2} and again $m = -1$, $n = 1$ we get

$$\frac{1}{1+x^2} = x^{-2} - x^{-4} + x^{-6} - x^{-8} + \dots \tag{2}$$

Newton said to use equation (1) if $x < 1$ and use equation (2) if $x > 1$. We know from more modern consideration that series (1) converges for $|x| < 1$ and series (2) converges for $|x^{-2}| < 1$ which is equivalent to $|x| > 1$.

Another way to obtain these two series is to do polynomial long division and divide $1 + x^2$ and $x^2 + 1$ into 1 respectively.

Calculation of the radius of curvature for a function $y = f(x)$ using the tools of Newton. We want to find a circle $x^2 + y^2 = r^2$ so that the circle is tangent to our function and has the same second derivative as our function. So this is equivalent to finding r in terms of the first and second derivative of the circle:

$$\begin{aligned}
y &= (r^2 - x^2)^{\frac{1}{2}} \\
y' &= \frac{1}{2}\Big(r^2 - x^2\Big)^{-\frac{1}{2}}(-2x) \\
&= -\Big(r^2 - x^2\Big)^{-\frac{1}{2}}x \\
&= \frac{-x}{y} && (3) \\
y'' &= \frac{1}{2}\Big(r^2 - x^2\Big)^{-\frac{3}{2}}(-2x^2) - \Big(r^2 - x^2\Big)^{-\frac{1}{2}} \\
&= \Big(r^2 - x^2\Big)^{-\frac{3}{2}}\Big(- x^2 - r^2 + x^2\Big) \\
&= \frac{-r^2}{(r^2 - x^2)^{\frac{3}{2}}} \\
&= \frac{-r^2}{y^3} && (4)
\end{aligned}$$

Using equation (3) and the fact that $x^2 = r^2 - y^2$ we have,

$$\begin{aligned}
y' &= \frac{-x}{y} \\
x &= -yy' \\
x^2 &= y^2(y')^2 \\
r^2 - y^2 &= y^2(y')^2 \\
r^2 &= y^2(y')^2 + y^2 \\
r^2 &= y^2((y')^2 + 1) \\
y^2 &= \frac{r^2}{(y')^2 + 1} \\
y &= \pm\sqrt{\frac{r^2}{(y')^2 + 1}}
\end{aligned}$$

Now we use equation (4):

$$\begin{aligned}
y'' &= \frac{-r^2}{y^3} \\
r^2 &= -y^3 y'' \\
r^2 &= \mp\left(\frac{r^2}{(y')^2 + 1}\right)^{\frac{3}{2}} y'' \\
r^2 &= \mp\frac{r^3 y''}{((y')^2 + 1)^{\frac{3}{2}}} \\
r &= \mp\frac{((y')^2 + 1)^{\frac{3}{2}}}{y''}
\end{aligned}$$

and since the radius must be positive, the sign from $\mp$ is taken according to whether or not y'' (that is y'' of our function) is positive.

Cauchy defined the center of curvature of a curve as the "intersection of two infinitely close normals" to the curve. The modern definition is obtained by using the normal and the radius of curvature defined above. The concept of curvature itself is defined (in modern terms) as the reciprocal of the radius of curvature - this gives the straight line a curvature of 0 (since "obviously" $\frac{1}{\infty} = 0$).