

Fundamental Theorem of Calculus

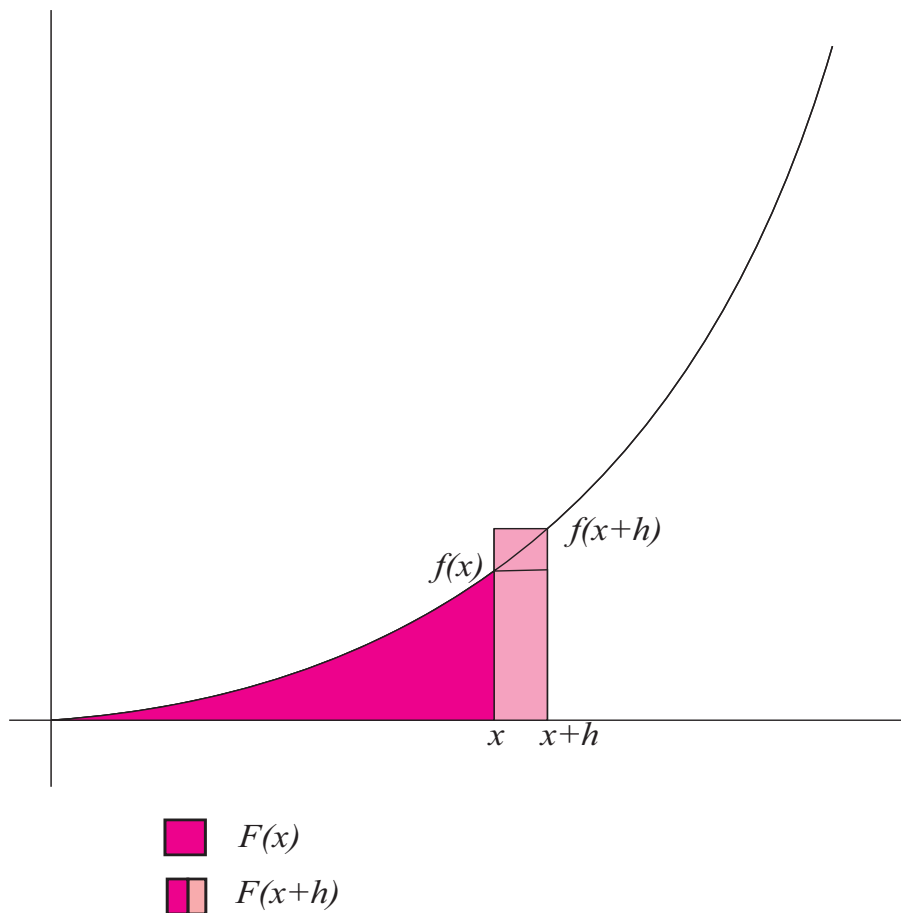


FIGURE 1. Fundamental Theorem of Calculus

Suppose that $y = f(x)$ is a function; for the sake of our picture I'll assume that it's an increasing function. We also assume that the function is continuous.

We want to calculate the area under the graph bounded by x -axis and the vertical lines through $x = a$ and $x = b$. We let $F(x)$ denote the area under the graph bounded by x -axis from $x = 0$ to the vertical line through x . This area is indicated in maroon. The area between the vertical lines at x and $x + h$ is indicated in pink; note that I've drawn the picture for the case of $h > 0$. So the area in pink is $F(x+h) - F(x)$. The area of the small rectangle (base \times height) is $h \cdot f(x)$ and is less than the area in pink which is less than the area of the big rectangle

$h \cdot f(x + h)$. So we have

$$h \cdot f(x) < F(x + h) - F(x) < h \cdot f(x + h).$$

Dividing by h gives us:

$$f(x) < \frac{F(x + h) - F(x)}{h} < f(x + h).$$

Using Newton's terminology, the ultimate ratio of the middle portion of the inequality is $F'(x)$; and of the right quantity is $f(x)$. In modern terminology we take the limit of the expression as $h \rightarrow 0$:

$$\lim_{h \rightarrow 0} f(x) < \lim_{h \rightarrow 0} \frac{F(x + h) - F(x)}{h} < \lim_{h \rightarrow 0} f(x + h).$$

In modern parlance the middle limit is the definition of the derivative and the rightmost limit follows from the continuity of the function f ; and so in any case our reasoning gives:

$$f(x) \leq F'(x) \leq f(x).$$

Leibnitz would have used the infinitesimal $h = dx$ and calculated as follows:

$$dx \cdot f(x) < dF < dx \cdot f(x + dx).$$

Using the "properties of infinitesimals," after dividing by dx , this becomes:

$$f(x) \leq \frac{dF}{dx} \leq f(x).$$

In either case we have $F'(x) = f(x)$. So F is an anti-derivative of f and in particular would be the one with $F(0) = 0$ (because the area from $x = 0$ to $x = 0$ is 0 and using the fact that two antiderivatives of a function only differ by a constant.) Then the area under the graph, above the x -axis and between the vertical lines through $x = a$ and $x = b$ (for $b > a$) would be:

$$\text{Area} = F(b) - F(a).$$