Lambert Quadrilaterals.

Definition. The quadrilateral $\square ABCD$ is called a *Lambert quadrilateral* if it has three right angles. [Notation, for the Lambert quadrilateral $\square ABCD$ the point D is usually assumed to be the angle which has not been designated as a right angle. (Though in Euclidean geometry it will be a right angle.)]

Definition. The quadrilateral $\square ABCD$ is called a *Saccheri quadrilateral* if has two congruent sides perpendicular to a third side, called the base of the quadrilateral. [Notation, for the Saccheri quadrilateral $\square ABCD$ the sidet $\overline{AB}$ is usually assumed to be the base with sides $\overline{DA}$ and $\overline{CB}$ perpendicular to it.]

Non-Euclidean Exercise: A study of quadrilaterals.

Claim 1. If $\square ABCD$ is a rectangle, then opposite side are congruent. [Note: this is a neutral geometry theorem.]

Claim 2. If $\square ABCD$ is a Saccheri quadrilateral with congruent sides $\overline{DA}$ and $\overline{CB}$, then the angles $\angle CDA$ and $\angle DCB$ are congruent.

Claim 3. Suppose $\square ABCD$ is a quadrilateral with right angles $\angle DAB$ and $\angle ABC$. Then the angle opposite the smaller side is smaller: if $m(\overline{DA}) < m(\overline{CD})$ then $m(\angle ADC) > m(\angle BCD)$.

Proof. Let E be chosen on $\overrightarrow{BC}$ so that $\overline{BE} \cong \overline{AD}$; since $m(AD) < m(BC)$ we have $B - -E - -D$.

$\angle ADE \cong \angle BED$, by Saccheri quadrilateral.

So $m(\angle ADE) < m(\angle ADC)$ since $\overrightarrow{DE}$ lies in the interior of $\angle ADC$.

$\angle BED > \angle BCD$, exterior angle theorem.

So $m(\angle ADC) > m(\angle BCD)$.

$\square$

Claim 4. Suppose $\square ABCD$ is a quadrilateral with right angles $\angle DAB$ and $\angle ABC$. Then the side opposite the larger angle is larger: if $m(\angle ADC) > m(\angle BCD)$ then $m(\overline{DA}) < m(\overline{CD})$.

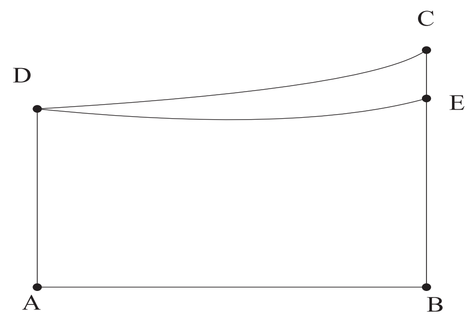

Figure 1:

Proof. If not we have a contradiction to claim 3.

□

Claim 5. Suppose ℓ and m are two parallel lines so that P and Q are points of ℓ whose distance from m are equal, then ℓ and m have a common perpendicular through the midpoint M of $\overline{PQ}$.

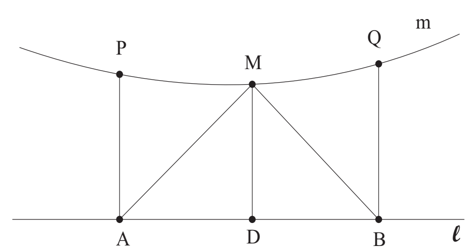

Figure 2:

Proof. Let M be the midpoint of $\overline{PQ}$ and let A, B, D be the bases of perpendicularity respectively from P, Q, M to ℓ.

$\overline{PA} \cong \overline{QB}$ by hypothesis;

$\angle APM \cong \angle BQM$, since $\square PABQ$ is a Saccheri quadrilateral;

$\overline{PM} \cong \overline{QM}$ since M is the midpoint;

so $\triangle MPA \cong \triangle MQB$ by SAS.
$\angle PMA \cong QMB$ congruencies.
$\overline{AM} \cong \overline{BM}$ congruencies;
$\overline{MB} \cong \overline{MB}$ identity;
$\angle MDA \cong MDB$ right angles;
$\triangle MDA \cong \triangle MDB$ SS -right angle.
$\angle AMD \cong \angle BMD$ congruencies;
$\angle PMD \cong \angle QMD$ angle addition of congruent angles.
Therefore $\overline{MD} \perp \overline{AB}$.
$\overline{AD} \cong \overline{BD}$ congruencies; therefore D is the midpoint of $\overline{AB}$. □

Corollary to Claim 5. The perpendicular bisector of the base of a Saccheri quadrilateral divides it into two Lambert quadrilaterals.

Claim 6. On the hypothesis of claim 5, every other point of ℓ is farther from m than M.

Proof. Referring to Figure 2: If $R \in m$ and X is the base of perpendicularity to ℓ then the quadrilateral $\square RXDM$ has right angles $\angle RXD, \angle XDM, \angle DMR$; since $def(\square RXDM > 0)$ it follows that $m(\angle MRX) < 90$. So the result follows from Claim 4. □

Claim 7. Suppose that ℓ and m are lines such that there is a segment PD with $P \in m$ and $D \in \ell$ so that PD is parallel to both ℓ and m. Then m and ℓ are perpendicular and if Q and R are points of m so that $QP \cong RP$ then Q and R are the same distance from ℓ.