Picture and proof related to Newton's lemmas.

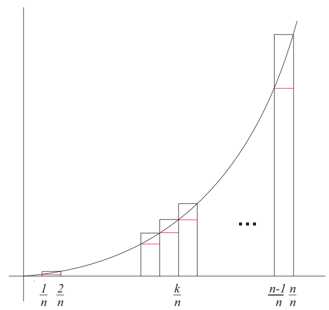

Figure 1: Upper and Lower Rectangles for $y = x^2$.

For the function $y = x^2$ we're interested in the area under the curve, above the x-axis and between $x = 0$ and $x = 1$. The interval $[0, 1]$ is partitioned into n subintervals each of length $\frac{1}{n}$. We will need the following formula.

$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}.$$

Let A denote the area under the parabola $y = x^2$ from $x = 0$ to $x = 1$. We know that the area is between the upper rectangles (in black) and the lower rectangles (in red). The area of the k^{th} upper rectangle is height $\times$ base and is $\left(\frac{k}{n}\right)^2 \times \frac{1}{n}$ and the k^{th} lower rectangle is $\left(\frac{k-1}{n}\right)^2 \times \frac{1}{n}$. Since the area is between these two collections of rectangles we have [I'm skipping some steps],

$$\begin{aligned} \sum_{k=1}^{n} \frac{(k-1)^2}{n^3} &< A < \sum_{k=1}^{n} \frac{k^2}{n^3} \\ \frac{(n-1)(n)(2n-1)}{6n^3} &< A < \frac{n(n+1)(2n+1)}{6n^3}. \end{aligned}$$

For all positive integers n so, by one of Newton's lemmas

$$\frac{1}{3} \leq A \leq \frac{1}{3}.$$