

Pascal's Triangle.

				1				
				1	1			
			1	2	1			
		1	3	3	1			
	1	4	6	4	1			
	1	5	10	10	5	1		
	1	6	15	20	15	6	1	
1	7	21	35	35	21	7	1	
		⋮			⋮			

I draw the triangle in its modern incarnation, its row shifted from Pascal's presentation. Pascal wrote what he had discovered about this configuration of numbers in his monograph *Triangle Arithmétique*. The top row is designated the 0th row, the zeroth row. The next one is the first row and so on. The elements are numbered from left to right (or from right to left - it doesn't make any difference because of the symmetry of the triangle - this is Consequence V in his book); and their position is labeled 0, 1, 2, ... and so on. So that the zeroth entry in each row is always a 1. For example, the second entry in the fifth row is 10; the first entry in the n^{th} is n . The modern notation for the r^{th} entry in the n^{th} row is $\binom{n}{r}$. The r^{th} entry of the $(n + 1)^{\text{th}}$ row of the triangle is calculated from the n^{th} row by adding the entries in the previous row that are north-west and north-east of it.

In closed form:

$$\binom{n}{r} = \frac{n!}{r!(n-r)!}.$$

In order for the formula to make sense when $r = 0$ or $r = n$ we use the notational convention that $0! = 1$ by definition. Following are identities for you to work on regarding the elements of Pascal's Triangle.

1. First we prove the correctness of the closed form representation for each integer $r = 0, 1, 2, \dots, n, n + 1$:

$$\binom{n+1}{r} = \binom{n}{r-1} + \binom{n}{r}.$$

Note: for $r = 0$ we assume $\binom{n}{-1} = 0$ and for $r = n + 1$ we assume $\binom{n}{n+1} = 0$. This is equivalent to assuming that there are 0's before the first and after the last element of each line.

$$\begin{array}{cccccccc}
 & & & & & & & 0 \\
 & & & & & & & 0 & 0 \\
 & & & & & & & 0 & 1 & 0 \\
 & & & & & & & 0 & 1 & 1 & 0 \\
 & & & & & & & 0 & 1 & 2 & 1 & 0 \\
 & & & & & & & 0 & 1 & 3 & 3 & 1 & 0 \\
 & & & & & & & 0 & 1 & 4 & 6 & 4 & 1 & 0 \\
 & & & & & & & 0 & 1 & 5 & 10 & 10 & 5 & 1 & 0 \\
 & & & & & & & & \vdots & & & & \vdots & &
 \end{array}$$

2. Proving the following identity shows why the term $\binom{n}{r}$ is called a binomial coefficient:

$$(a + b)^n = \sum_{r=0}^n \binom{n}{r} a^{n-r} b^r.$$

3. In Pascal's monograph, Consequence II is the following:

$$\binom{n}{r} = \sum_{i=1}^{n-r+1} \binom{n-i}{r-1}.$$

4.) Pascal's Consequence VIII calculates the sum of all the elements in a row:

$$\sum_{r=0}^n \binom{n}{r} = 2^n.$$

5.) Pascal's Consequence XII is of particular historical significance; it is the first time that induction in the modern formulation was stated and used to prove a statement; using the modern formulation it says that for all positive integers n and $r = 1, 0, \dots, n - 1$:

$$\frac{\binom{n}{r}}{\binom{n}{r+1}} = \frac{r+1}{n-r}.$$

6.) Finally, for an application to probability: show that the probability of getting exactly r sixes in n throws of a fair die is:

$$\text{Prob} = \binom{n}{r} \left(\frac{1}{6}\right)^r \left(\frac{5}{6}\right)^{n-r}.$$