The Poincaré Model for the Hyperbolic Plane Geometry.

Let E^2 denote the Cartesian plane. Note that we will also think of it as the complex plane for some calculations.

Let $\mathcal{D} = \{(x,y)|x^2+y^2 < 1\}$; equivalently if we think of (x,y) as the complex number $z = x+yi$, then $\mathcal{D} = \{z \big| \ |z| < 1\ \}$. The points of $\mathcal{D}$ are the points in our hyperbolic geometry. So the points of this model is a subset of Cartesian or Complex plane.

Define $S^1 = \{(x,y)|x^2+y^2 = 1\}$ or equivalently, $S^1 = \{z \big| \ |z| = 1\ \}$.

Definition of a line in the model: ℓ is a line if and only if 1. ℓ is the common part of $\mathcal{D}$ and a usual line in E^2 that contains the origin $(0,0)$, or 2. ℓ is the common part of $\mathcal{D}$ and a circle perpendicular to S^1 at each intersection.

Definition. The angle between two intersecting circles is the angle between their tangent lines at the intersection point. Similarly, the angle between a line and a circle is the angle between the line and the tangent line to the circle at the point.

Definition. If w and z are two points in $\mathcal{D}$ (thought of as points in the complex plane) then the measure of the line segment $\overline{wz}$ in our model is defined as:

$$m(\overline{wz}) = \ln\Big(\frac{|1-z\overline{w}| + |z-w|}{|1-z\overline{w}| - |z-w|}\Big)$$

where on the right side of the equation z and w are interpreted as complex numbers and $\overline{w}$ denotes the complex conjugate of w.

Theorem. The Poincaré model satisfies the axioms of geometry with the hyperbolic axiom [the acute angle axiom for the Saccheri or Lambert quadrilaterals].

Poincaré's Universe.

Main assumptions are equivalent to the following: The Universe consists of the points inside a sphere of radius R such that the length of a measuring rod at distance r from the center is given by $\ell = k(R^2 - r^2)$. We let $k = 1$ as our standard unit of measurement. Since the distance $d(A, B)$ between two objects A and B is measured by seeing how many times the standard unit divides into the distance $d(A.B)$, it follows that the length of a rod from r to $r + \Delta r$ will be approximately $\frac{\Delta r}{R^2 - r^2}$. So then the length $\ell(r_1, r_2)$ of a path along a radial ray from r_1 to r_2 will be:

$$\begin{aligned}
\ell(r_1, r_2) &= \int_{r_1}^{r_2} \frac{1}{R^2 - r^2} dr \\
&= \int_{r_1}^{r_2} \Big(\frac{\frac{1}{2R}}{R + r} + \frac{\frac{1}{2R}}{R - r} \Big) dr \\
&= \frac{1}{2R} (\ln(R + r) - \ln(R - r)) \Big|_{r_1}^{r_2} \\
&= \frac{1}{2R} \Big(\ln\Big(\frac{R + r_2}{R - r_2}\Big) - \ln\Big(\frac{R + r_1}{R - r_1}\Big) \Big) \\
&= \frac{1}{2R} \ln\Big(\frac{(R + r_2)(R - r_1)}{(R - r_2)(R + r_1)}\Big).
\end{aligned}$$

So the radius of the Universe would appear to be:

$$\begin{aligned}
\int_0^R \frac{1}{R^2 - r^2} dr &= \frac{1}{2R} \ln\Big(\frac{(R + R)R}{(R - R)R}\Big) \\
&\to \infty.
\end{aligned}$$