Saccheri Quadrilaterals.

Definition. A Saccheri quadrilateral is a quadrilateral with two congruent sides perpendicular to a third side, called the base of the quadrilateral.

Exercises: In working with these you may assume the the SAS axiom and the ASA and SSS theorems.

Given: For $\square ABCD$ we have $\overline{AD} \cong \overline{BC}$, angles $\angle ABC$ and $\angle BAD$ are congruent right angles.

Exercise 1. *The summit angle of a Saccheri Quadrilateral are equal.*

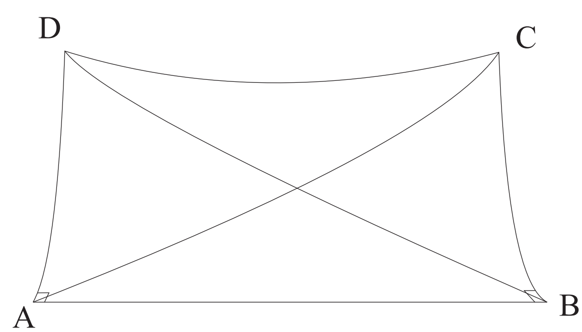

Figure 1: Sccheri Quadrilateral.

Proof.

$$\begin{aligned} \triangle ABC &\cong \triangle BAD \text{ (SAS)} \\ \overline{AC} &\cong \overline{BD} \\ \triangle ADC &\cong \triangle BCD \text{ (SSS)} \\ \therefore \angle ADC &\cong \angle BCD \end{aligned}$$

□

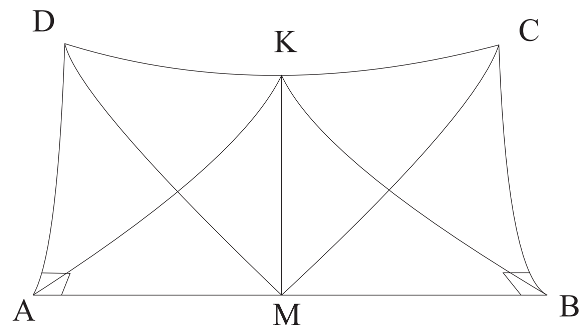

Figure 2: Saccheri Quadrilateral - Midpoint Theorem.

Exercise 2. *The line joining the midpoint of the base and summit of a Saccheri Quadrilateral is perpendicular to both of them.*

Proof.

$$\begin{aligned}
\triangle MAD &\cong \triangle MBC \text{ (SAS)} \\
\overline{MD} &\cong \overline{MC} \\
\triangle MDK &\cong \triangle MCK \text{ (SSS)} \\
\angle MKD &\cong \angle MKC \text{ (and therefore is a right angle)} \\
\angle ADK &\cong \angle BCK \text{ (exercise 1)} \\
\triangle ADK &\cong \triangle BCK \text{ (SAS)} \\
\overline{AK} &\cong \overline{BK} \\
\triangle AMK &\cong \triangle BMK \text{ (SSS)} \\
\angle AMK &\cong \angle BMK \text{ (and therefore is a right angle)}
\end{aligned}$$

□

Exercise 3. *If perpendiculars are drawn from the extremities of the base of a triangle upon the line passing through the midpoints of the two sides, then a Saccheri quadrilateral is formed.*

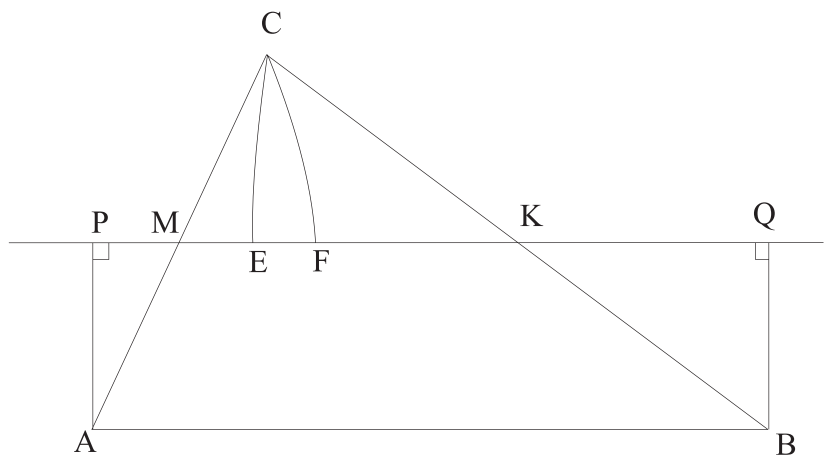

Figure 3: Triangle theorem.

Proof. Given:

$$\begin{array}{rcl} \overline{AM} & \cong & \overline{CM} \\ \overline{BK} & \cong & \overline{CK} \\ \overline{AP} & \perp & \overleftrightarrow{MK} \\ \overline{BQ} & \perp & \overleftrightarrow{MK} \end{array}$$

Construct E so that $\overline{PM} \cong \overline{EM}$ and F so that $\overline{QK} \cong \overline{FK}$ (in the case that

E and F are in the reverse order, the proof is the same). Then:

$$\begin{aligned}
\angle AMP &\cong \angle CME \text{ (vertical angles)} \\
\overline{AM} &\cong \overline{CM} \\
\overline{PM} &\cong \overline{EM} \text{ (construction)} \\
\triangle APM &\cong \triangle CEM \text{ (SAS)} \\
\angle APM &\cong \angle CEM \text{ and so is a right angle} \\
\angle BKQ &\cong \angle CKF \text{ (vertical angles)} \\
\overline{BK} &\cong \overline{CK} \\
\overline{QK} &\cong \overline{FK} \text{ (construction)} \\
\triangle BQK &\cong \triangle CFK \text{ (SAS)} \\
\angle BQK &\cong \angle CFK \text{ and so is a right angle} \\
\triangle CEF &\cong \triangle CFE \text{ (ASA)} \\
\overline{CE} &\cong \overline{CF} \\
\overline{AP} \cong \overline{CE} &\cong \overline{CF} \cong \overline{BQ}
\end{aligned}$$

So $\square PQBA$ is a Saccheri quadrilateral. □

Exercise 4. *The line joining the midpoints of the equal sides of a Saccheri quadrilateral is perpendicular to the line joining the midpoints of the base and summit.*

Proof. Since $\overline{AD} \cong \overline{BC}$ and E and F bisect $\overline{AD}$ and $\overline{BC}$ respectively we have:

$$\overline{AE} \cong \overline{BF}$$

and since M is the midpoint of AB

$$\overline{AM} \cong \overline{BM}$$

So:

$$\begin{aligned}
\triangle EAM &\cong \triangle FBM \text{ (SAS)} \\
\angle AME &\cong \angle BMF \\
\overline{EM} &\cong \overline{FM}
\end{aligned}$$

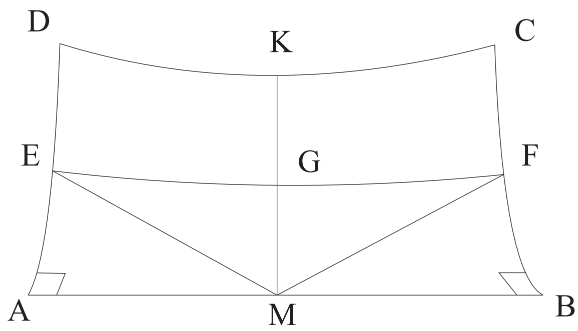

Figure 4: Midline theorem.

Since, by exercise 2, $\overline{KM} \perp \overline{AB}$ then "subtracting" congruent pieces we have:

$$\begin{aligned} \angle EMG &\cong \angle FMG \\ \triangle EMG &\cong \triangle FMA \text{ (SAS)} \\ \angle MGE &= \angle MGF \text{ (and so is a right angle).} \end{aligned}$$

□