Solving The Cubic Equation

Following is the technique that Tartaglia and Ferro used to derive the solutions to the cubic equation. The general cubic equation can easily be reduced to the following equation (by dividing by the x^3 coefficient if needed.)

$$x^3 + ax^2 + bx + c = 0. \tag{1}$$

First Tartaglia solved an easier cubic:

$$x^3 + px = q. \tag{2}$$

He used the following identity which is obtained by expanding $(r-s)^3$:

$$(r-s)^3 + 3rs(r-s) = r^3 - s^3$$

If we use the following substitutions, this identity can be made to look like equations (2) above:

$$\begin{aligned} x &= r - s \\ p &= 3rs \\ q &= r^3 - s^3. \end{aligned} \tag{3}$$

Which is two equations in the unknowns r and s in terms of the known p and q; so we can solve for them:

$$\begin{aligned} q^2 &= r^6 - 2r^3s^3 + s^6 \\ p^3 &= 27r^3s^3 \\ \frac{4p^3}{27} &= 4r^3s^3 \end{aligned}$$

Add the third to the first equation gives us a perfect square:

$$\begin{aligned} q^2 + \frac{4p^3}{27} &= r^6 + 2r^3s^3 + s^6 \\ \sqrt{q^2 + \frac{4p^3}{27}} &= r^3 + s^3. \end{aligned}$$

So with equation (3) above we now have the following from which r and s can easily be solved

$$\begin{aligned} r^3 + s^3 &= \sqrt{q^2 + \frac{4p^3}{27}} \\ r^3 - s^3 &= q. \end{aligned}$$

Solving for r and s by adding and then subtracting these two:

$$\begin{aligned} 2r^3 &= \sqrt{q^2 + \frac{4p^3}{27}} + q \\ 2s^3 &= \sqrt{q^2 + \frac{4p^3}{27}} - q \\ r &= \sqrt[3]{\frac{1}{2}\Big(\sqrt{q^2 + \frac{4p^3}{27}} + q\Big)} \\ s &= \sqrt[3]{\frac{1}{2}\Big(\sqrt{q^2 + \frac{4p^3}{27}} - q\Big)}. \end{aligned}$$

And now we can obtain $x = r - s$:

$$x = \sqrt[3]{\frac{1}{2}\Big(\sqrt{q^2 + \frac{4p^3}{27}} + q\Big)} - \sqrt[3]{\frac{1}{2}\Big(\sqrt{q^2 + \frac{4p^3}{27}} - q\Big)}.$$

Now he would want to convert equation (1) into equation (2). Let's rewrite equation (1) in terms of t rather than x:

$$t^3 + at^2 + bt + c = 0 \tag{4}$$

and perform the following substitution: $t = x - d$. We want to find a value of d that makes the a term "disappear."

$$\begin{aligned} t^3 + at^2 + bt + c &= (x-d)^3 + a(x-d)^2 + b(x-d) + c \\ &= (x^3 - 3x^2d + 3xd^2 - d^3) + a(x^2 - 2xd + d^2) + b(x-d) + c \\ &= x^3 + (a - 3d)x^2 + (b - 2ad + 3d^2)x + (c - bd + ad^2 - d^3) \end{aligned}$$

We see that letting $d = \frac{a}{3}$ makes the a term "disappear.

So to solve the equation $t^3 + at^2 + bt + c = 0$ you let $x = t - \frac{a}{3}$ to obtain equation (2) with

$$\begin{aligned} p &= b - 2ad + 3d^2 \\ &= b - 2a\frac{a}{3} + 3\left(\frac{a}{3}\right)^2 \\ q &= -(c - bd + ad^2 - d^3) \\ &= -c + b\frac{2}{3} - a\left(\frac{a}{3}\right)^2 + \left(\frac{a}{3}\right)^3. \end{aligned}$$

Then we solve the equation $x^3 - px = q$ using Tartaglia's technique above and once we have that solution we obtain t by letting $t = x - \frac{a}{3}$.