Spherical Trigonometry.

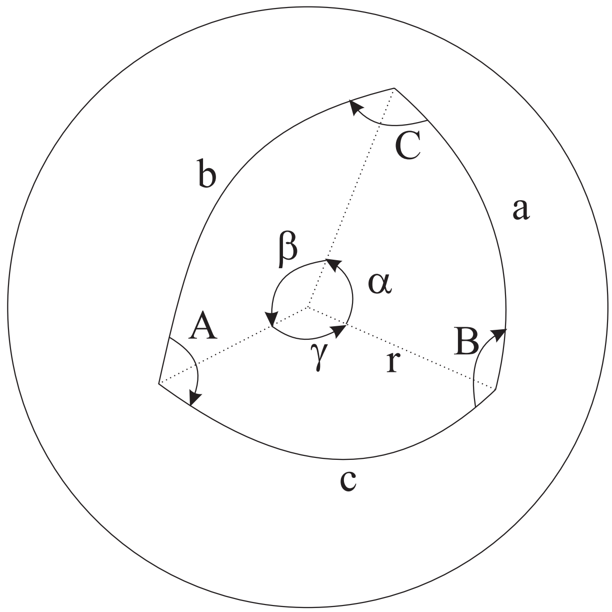

Figure 1: Trigonometry on a sphere of radius r.

The sine rules:

$$\frac{\sin A}{\sin \alpha} = \frac{\sin B}{\sin \beta} = \frac{\sin C}{\sin \gamma}.$$

Or equivalently:

$$\frac{\sin A}{\sin \frac{a}{r}} = \frac{\sin B}{\sin \frac{b}{r}} = \frac{\sin C}{\sin \frac{c}{r}}. \tag{1}$$

so that for a sphere of radius $r = 1$ we have:

$$\frac{\sin A}{\sin a} = \frac{\sin B}{\sin b} = \frac{\sin C}{\sin c}.$$

Cosine laws:

$$\begin{aligned}
\cos\alpha &= \cos\beta\cos\gamma + \sin\beta\sin\gamma\cos A \\
\cos\beta &= \cos\gamma\cos\alpha + \sin\gamma\sin\alpha\cos B \\
\cos\gamma &= \cos\alpha\cos\beta + \sin\alpha\sin\beta\cos C.
\end{aligned}$$

Now consider equation (1) as $r \to \infty$:

$$\frac{\sin A}{\sin\frac{a}{r}} = \frac{\sin B}{\sin\frac{b}{r}} = \frac{\sin C}{\sin\frac{c}{r}}$$

$$\frac{\sin A}{r\sin\frac{a}{r}} = \frac{\sin B}{r\sin\frac{b}{r}} = \frac{\sin C}{r\sin\frac{c}{r}}$$

$$\lim_{r\to\infty}\frac{\sin A}{r\sin\frac{a}{r}} = \lim_{r\to\infty}\frac{\sin B}{r\sin\frac{b}{r}} = \lim_{r\to\infty}\frac{\sin C}{r\sin\frac{c}{r}}$$

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$

which is the usual law of sines for (Euclidean) plane trigonometry.

The critical angle.

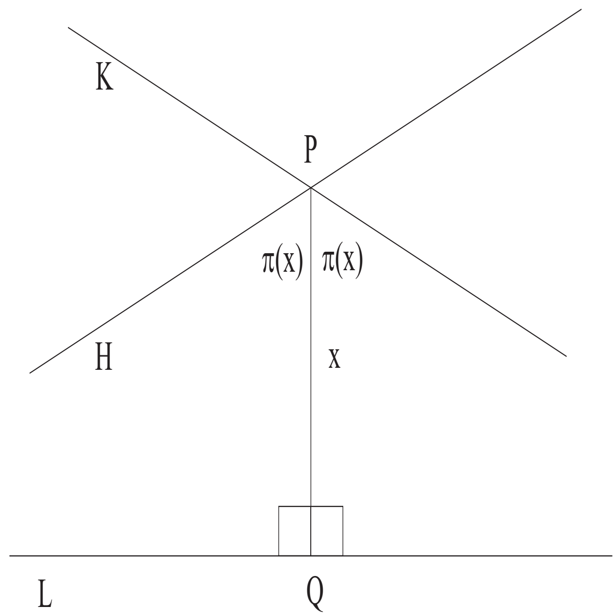

Figure 2: The critical angle for a line segment of length x.

Consider a point P and a line L not containing P. Let $\overline{PQ}$ be the Perpendicular from P to L intersecting L at the point Q. Let x be the length of $\overline{PQ}$. Then for all the lines containing P and not intersecting L there will be two lines, H and K both making angles $\pi(x)$ with $\overline{PQ}$ but on opposite sides of $\overline{PQ}$ so that any line ℓ containing P making an angle less than $\pi(x)$ with $\overline{PQ}$ will intersect L. It is not difficult to argue that the lines H and K themselves will not intersect L.

Lobatchevsky and Gauss (the latter in an unpublished letter) show that

$$\tan\left(\frac{\pi(x)}{2}\right) = e^{-\frac{x}{k}}$$

where k is a constant dependent upon the unit of length used. Observe that as $x \to 0$, $\pi(x) \to \frac{\pi}{2}$. Assuming $k = 1$ (which would correspond to sphereical trigonometry on a sphere of radius $r = 1$) , Lobatchevsky obtained the following trigonometric like formulas for the trigonometry of his geometry:

$$\begin{aligned}\sin\alpha &= \cos\beta\sin(\pi(b))\\ \sin\pi(c) &= \sin\pi(a)\sin\pi(b)\\ \cot\pi(a) &= \cot\pi(c)\sin\alpha.\end{aligned}$$

He further observed that these formulas correspond to spherical trigonometric formulas but on a sphere with radius $r = i = \sqrt{-1}$.

Claim: The last of these formulas can be converted to

$$\sinh a = \sinh b \sin\alpha.$$

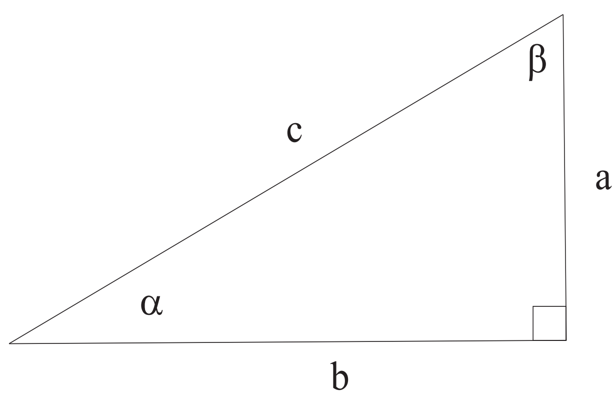

Figure 3: Non-Euclidean triangle.