Neutral Geometry.

The neutral plane geometry, the geometry without a parallel line axiom, is built up of some incidence/containment axioms, betweenness axioms, congruency axioms and one important axiom about triangles. (These axioms are in addition to the standard assumptions of logic.) Undefined quantities include: points, lines, sets, etc. ... The following are sample axioms and are not meant to be all inclusive.

Incidence Axioms:
1: For each pair of points there is a unique line containing them.
2: Every line contains at least two points.
3: There exist three non co-linear points.

Betweenness axioms.
1: If the point P is between the points A and B then they are co-linear.
2: Given two points P and Q there exist three points A, B and C so that:
P is between A and Q;
C is between P and Q;
Q is between P and C.
3: Given three points contained in a line, exactly one is between the other two.

An axiom about a line separating the plane.

The congruency axioms tell us that congruent line segments can be found on different lines with certain (intuitive) properties and that there exist congruent angles with sides on the same or different lines with certain properties.

Finally the SAS condition is assumed by axiom: Given two triangles with corresponding sides and angles so that two sides with the included angle of one triangle is congruent to the corresponding sides and included angle of the other triangle, then the triangles themselves are congruent.

From these axioms the following well known theorems follow.

Theorem [ASA]: Given two triangles with corresponding sides and angles so that two angles with the side between them of one triangle is congruent to the two corresponding angles with the side between them of the other triangle, then the triangles themselves are congruent.

Theorem [SSS]: Given two triangles with corresponding sides and angle so that three of the corresponding sides of one triangle are congruent to the three corresponding sides of the other triangle, then the triangles themselves are congruent.

Theorem [Alternate Interior Angle Theorem]: Given two lines cut by a transversal so that the alternate interior angles are congruent, then the lines are parallel.

(Observe that this theorem implies that an Elliptic type geometry - such as the geometry on the sphere - must have a different set of axioms.)

Exterior Angle Theorem and Implications.

Some implications of the neutral geometry with proofs.

Exterior Angle Theorem. If $\triangle ABC$ is a triangle and D is a point on the line $\overleftrightarrow{BC}$ so that C is between B and D, then $m(\angle BAC) < m(\angle ACD)$ and $m(\angle ABC) < m(\angle ACD)$.

Proof.

Suppose the hypothesis of the theorem and suppose that the theorem is not true and that $m(\angle ACD) \leq m(\angle BAC)$. If $m(\angle ACD) = m(\angle BAC)$ then lines $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$ are parallel by the congruency of alternate interior angles theorem. This would imply that we don't have a triangle. So it follows that if the theorem is not true that we must have: $m(\angle ACD) < m(\angle BAC)$. By Axiom there is a point P on the opposite side of $\overleftrightarrow{AC}$ than D so that $\angle CAP \cong \angle ACD$.

Since $m(\angle ACD) < m(\angle BAC)$ then by the measurement of angles property, $P \in Int(\angle BAC)$. So by the crossbar theorem $\overrightarrow{AP}$ intersects $\overline{BC}$ at some point E. But $m(\angle ACD) = m(\angle CAP)$ implies that $\angle ACD \cong \angle CAP$, so again by the congruency of alternate interior angles theorem $\overleftrightarrow{AE}$ is parallel to $\overleftrightarrow{CD}$ and that contradicts the fact that those lines are supposed to intersect at the point E. Thus the theorem is true. $\square$

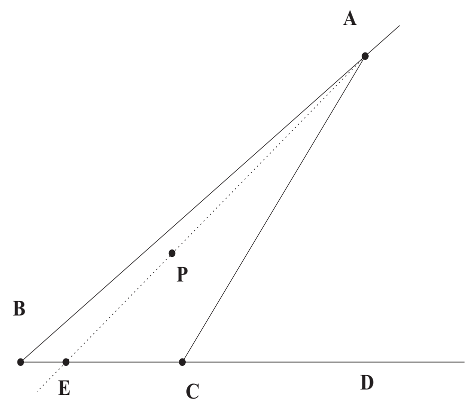

Figure 1: Exterior Angle Theorem

Corollary. The sum of the measures of two angles of a triangle is less than $180°$.

Lemma 1 to Saccheri-Legendre Theorem (Midpoint Theorem). If $\overline{AC}$ is a line segment then there is a point B on $\overline{AC}$ so that $A--B--C$ and $\overline{AB} \cong \overline{BC}$.

Proof.

Let $\overline{AC}$ be the line segment. Let D be a point not on $\overline{AC}$, let Q be a point on the opposite side of $\overline{AC}$ as D so that $\angle ACQ \cong \angle CAD$. Let E be a point on $\overrightarrow{CQ}$ so that $\overline{CE} \cong \overline{AD}$. Then since E and D are on opposite sides of $\overline{AC}$ there is a point $B \in \overleftrightarrow{AC}$ so that $D--B--E$. We have:

$\overleftrightarrow{AD}$ and $\overleftrightarrow{CE}$ are parallel by equal alternate interior angles;
D and A are on the same side of $\overleftrightarrow{CE}$ by parallel lines;
D and B are on the same side of $\overleftrightarrow{CE}$ since $D--B--E$;
A and B are on the same side of $\overleftrightarrow{CE}$.

Similarly:

C and E are on the same side of $\overleftrightarrow{AD}$ by parallel lines;

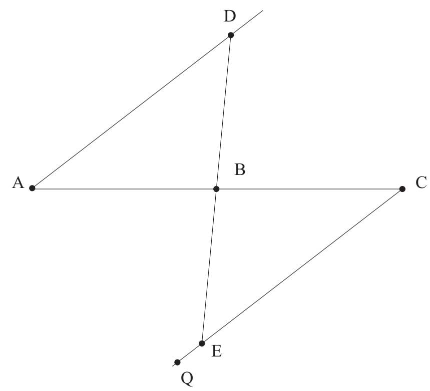

Figure 2: Midpoint theorem

E and B are on the same side of $\overleftrightarrow{AD}$ since $D--B--E$;
C and B are on the same side of $\overleftrightarrow{AD}$.
Therefore $A--B--C$.

Thus we have:
$\angle ACE \cong \angle CAD$ by construction;
$\angle ABD \cong CBE$ vertical angles;
$\overline{AD} \cong \overline{CE}$ by construction.
So $\triangle ABD \cong \triangle CBE$ by AAS.
$\overline{AB} \cong \overline{BC}$ by congruency of triangles. $\square$

Lemma 2 to Saccheri-Legendre Theorem. If $\triangle ABC$ is a triangle then there exists a triangle $\triangle PQR$ so that $m(\angle RPQ) \leq \frac{1}{2}m(\angle ABC)$ and so that the sum of the measures of $\triangle ABC$ is the same as the sum of the measures of $\triangle PQR$.

Proof.
Suppose $\triangle ABC$ is a triangle as in the hypothesis. Let D be the midpoint of $\overline{AC}$; thus $\overline{AD} \cong \overline{CD}$. Let E be a point on the ray $\overrightarrow{BD}$ on the other side of $\overleftrightarrow{AC}$ than B so that $\overline{BD} \cong \overline{DE}$. Then:
$\overline{BD} \cong \overline{ED}$ by construction;
$\angle BDA \cong \angle EDC$ vertical angles;
$\overline{AD} \cong \overline{CD}$ by midpoint theorem.
$\triangle BDA \cong \triangle EDC$ by SAS.

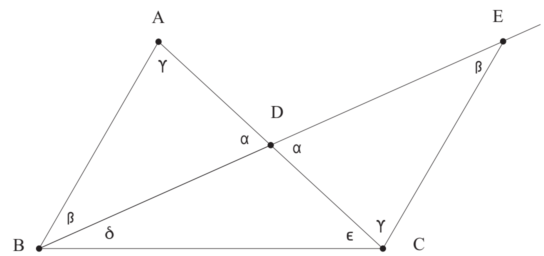

Figure 3: Lemma 2 to Saccheri-Legendre Theorem

$\angle BAD \cong \angle ECD$ congruency of triangles;
$\angle ABD \cong \angle CED$ congruency of triangles.

Let: $\beta = m(\angle DEC) = m(\angle DBA)$, $\gamma = m(\angle BAD) = m(\angle ECD)$, $\delta = m(\angle DBC)$ and $\epsilon = m(\angle BCA)$. Then $m(\angle ABC) = \beta + \delta$ so one of β or δ is less than or equal to $\frac{1}{2}m(\angle ABC)$.

Let the triangle $\triangle PQR$ be the triangle $\triangle BCA$. If $\beta \leq \frac{1}{2}m(\angle ABC)$ then $\angle BEC$ is the required angle; if $\delta \leq \frac{1}{2}m(\angle ABC)$ then $\angle EBC$ is the required angle. □

Saccheri-Legendre Theorem. The sum of the measures of the angles of a triangle is not greater than 180°.

Proof. Suppose that the theorem is not true. Then there is a triangle $\triangle ABC$ so that the sum of the angle measures is greater than 180°. Let: $\alpha = m(\angle ABC)$, $\beta = m(\angle BCA)$ and $\gamma = m(\angle CAB)$ and suppose $\alpha+\beta+\gamma > 180$. Let $\epsilon = 180 - \alpha - \beta - \gamma$. Repeated use of lemma 2 produces a sequence of triangles $\triangle P_nQ_nR_n$ so that for each n the sum of the angle measurements is same as for $\triangle ABC$, namely: $\alpha + \beta + \gamma$ but so that $m(\angle P_nQ_nR_n) \leq (\frac{1}{2})^n\alpha$. By the Archimedes Axiom there is an integer n so that $(\frac{1}{2})^n\alpha < \epsilon$, but then the sum of the measures of the other two angles of $\triangle P_nQ_nR_n$ would exceed 180° which contradicts the corollary to the exterior angle theorem. □

Non-Euclidean Geometry Notes.

Axiom HY [Hyperbolic Parallel Line Axiom] There is a line ℓ and a point P not on ℓ so that there are (at least) two distinct lines containing P and parallel to ℓ.

Definition. A neutral geometry that satisfies Axiom HY is called a *hyperbolic Geometry.*

For the following theorems assume that the axioms of hyperbolic geometry hold.

Theorem. There exists a triangle whose sum of angles is less than 180.

Theorem. There are no rectangles.

Theorem. The sum of the angles of a quadrilateral is less than 360.

Theorem. [AAA] Suppose that $\triangle ABC$ and $\triangle PQR$ are triangles such that $\angle ABC \cong \angle PQR$, $\angle BCA \cong \angle QRP$ and $\angle CAB \cong \angle RPQ$. Then $\triangle ABC \cong \triangle PQR$.

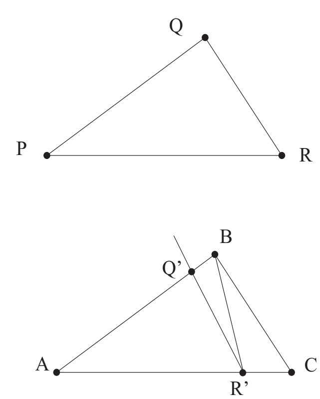

Figure 4: AAA Theorem

Proof. Suppose that the theorem is not true and that $\triangle ABC$ and $\triangle PQR$ are triangles such that $\angle ABC \cong \angle PQR$, $\angle BCA \cong \angle QRP$ and $\angle CAB \cong \angle RPQ$, but that the triangles are not congruent. We note that although not congruent, $def(\triangle PQR) = def(\triangle ABC)$. Then it must be the case that some pair of sides are not congruent or else the triangles would be congruent by ASA. So wlog assume that $\overline{PQ} \not\cong \overline{AB}$ and that $m(\overline{PQ}) < m(\overline{AB})$. Then let $Q' \in \overline{AB}$ be such that $\overline{AQ'} \cong \overline{PQ}$ so we have $A - -Q' - -B$. Let $R' \in \overrightarrow{AC}$ be such that $\overline{AR'} \cong \overline{PR}$. Then $\triangle AQ'R' \cong \triangle PQR$ by SAS. So $\angle AQ'R' \cong \angle ABC$, then $\overleftrightarrow{Q'R'} \parallel \overleftrightarrow{BC}$ by vertical angles and equal alternate interior angles, so R' must lie between A and C.

But then:

$def(\triangle ABC) = def(\triangle AQ'P') + def(\triangle Q'R'B) + def(\triangle R'BC) > def(\triangle AQ'R') = def(\triangle PQR)$.

Which is a contradiction. $\square$

Equivalences to the Parallel Line Axiom.

Following are some statements, which hold in the Euclidean Geometry, which turn out to be equivalent to the Euclidean Parallel line postulate:

- The sum of the angles of a every triangle is $180°$.
- The sum of the angles of at least one triangle is $180°$.
- The Pythagorean Theorem.
- There exists a rectangle.
- There exist two non-congruent similar triangles.