Ancient Egyptian Fractions

The ancient Egyptians, at least by the time of the Rhind Papyrus, $\sim$1,550 BCE, had a special symbol for $\frac{2}{3}$ and otherwise had a symbol that indicated the reciprocal of an integer. Consequently, they expressed fractions as a sum of reciprocals of integers (plus $\frac{2}{3}$ as needed.) I'll call these type of fractions *unit fractions*.

The Rhind Papyrus has sample problems about fractions and the representation of them as integers. If an Egyptian is going to do some arithmetic with fractions, adding two fractions in their representation is going to make the problem of expressing a fraction with numerator 2 as a sum of unit fractions particularly important. Although it is not certain how they obtained their expressions, following, in modern notations, are some helpful rules. [Exercise: show that the this representation is not unique.] Following are helpful rules for expressing a fraction as a sum of unit fractions (which you should prove for yourself):

$$\frac{2}{3n} = \frac{1}{3n} + \frac{1}{6n}. \tag{1}$$

Rule (1) is only useful if the denominator is a multiple of 3.

$$\frac{1}{n} = \frac{1}{n+1} + \frac{1}{n(n+1)}. \tag{2}$$

Rule (2) allows one to obtain the following calculation:

$$\frac{2}{n} = \frac{1}{n} + \frac{1}{n+1} + \frac{1}{n(n+1)}.$$

Around 1200, Fibonacci proved that every rational number between 0 and 1 can be expressed as a sum of unit fractions. I'll outline his argument, you should fill in the details. So we consider a fraction $\frac{a}{b}$ and first find an integer m so that

$$\frac{1}{m} \leq \frac{a}{b} \leq \frac{1}{m-1}.$$

Then let

$$\frac{c}{d} = \frac{a}{b} - \frac{1}{m}.$$

Use this fact to show that there exist integers m', c', d' so that

$$\frac{a}{b} = \frac{1}{m} + \frac{1}{m'} + \frac{c'}{d'}.$$

where

$$m' < m \quad \text{and} \quad c' < c.$$

Then complete the argument by induction until $c' = 1$.