Eudoxus and the Theory of Proportions.

There is evidence that Euclid's fifth book is based on Eudoxus' correction to the theory of proportions as constructed by the Pythagoreans before the discovery of incommensurable (irrational) quantities. According to the translation in Wikipeda, Euclid (Book 5 Definition 5) defines two things to be in the same proportion as two others when:

"Magnitudes are said to be in the same ratio, the first to the second and the third to the fourth when, if any equimultiples whatever be taken of the first and third, and any equimultiples whatever of the second and fourth, the former equimultiples alike exceed, are alike equal to, or alike fall short of, the latter equimultiples respectively taken in corresponding order."

This translates to the following: $a : b :: c : d$ written in modern notation is:

$$\frac{a \text{ (first)}}{b \text{ (second)}} = \frac{c \text{ (third)}}{d \text{ (fourth)}}.$$

The definition tells us that if (in modern notation) $\frac{a}{b}$ and $\frac{c}{d}$ are two proportions (ratios) then $\frac{a}{b} = \frac{c}{d}$ means that if m and n are two integers then all the following hold:

$$ma < nb \quad \Leftrightarrow \quad mc < nd \tag{1}$$

$$ma = nb \quad \Leftrightarrow \quad mc = nd \tag{2}$$

$$ma > nb \quad \Leftrightarrow \quad mc > nd. \tag{3}$$

(Where the symbol $\Leftrightarrow$ means that the left side implies the right side and the right side implies the left side; one can also say "if and only if.") Recall that the Greeks did not have a concept of negative numbers (nor of zero as a number) so all these quantities are positive.

There are two main modern presentations of the real numbers:

(1) First we assume the axioms of the integers, then build up the rationals from the integers then build up the irrationals from the rational by a completeness condition, such as the Dedekind cut property; or

(2) Axiomatically where the Dedekind cut property is replaced by a completeness axiom, like the least upper bound axiom.

In modern parlance, for example, one needs to use the completeness property to prove that there is a solution to the equation $x^2 = 2$. The Euclidean definition is a type of completeness axiom.

Now I'd like to prove that the Euclidean (Eudoxean) definition works. First we prove that if $\frac{a}{b} = \frac{c}{d}$ then conditions (1) through (3) hold. So suppose we have the left side of condition (1),

$$ma \quad < \quad nb.$$

Then $\frac{a}{b} = \frac{c}{d}$ implies that

$$a \quad = \quad \frac{cb}{d};$$

then

$$\begin{aligned} ma < nb \quad &\Rightarrow \quad m\frac{cb}{d} < nb \\ &\Rightarrow \quad m\frac{c}{d} < n \\ &\Rightarrow \quad mc < nd. \end{aligned}$$

Which is the right side of condition (1). The other (five) implications are similarly proven.

Next I'd like to prove that conditions (1) - (3) of the definition imply that $\frac{a}{b} = \frac{c}{d}$. To do this we suppose that the conditions hold but that $\frac{a}{b} \neq \frac{c}{d}$; this assumption will lead to a contradiction thus proving the statement. So suppose, without loss of generality, that $\frac{a}{b} < \frac{c}{d}$. Then there is a rational fraction, which is the ratio of two integers n and m so that

$$\frac{a}{b} < \frac{n}{m} < \frac{c}{d}.$$

Then:

$$\begin{array}{ccc} \frac{a}{b} < \frac{n}{m} & \text{and} & \frac{n}{m} < \frac{c}{d} \\ ma < nb & \text{and} & nd < mc \\ ma < nb & \text{and} & mc > nd.. \end{array}$$

But this contradicts condition (1) of the definition. So the statement is true. Observe that what I did was to find two specific integers n and m where $\frac{a}{b} \neq \frac{c}{d}$ implied that the condition didn't hold for these two integers.