Euler's manipulation of infinite series.

Euler first published this technique in 1735; but, apparently aware that he was making some unwarranted assumptions, he published a proof that his result was correct in 1741. It took another 100 years for the concept of infinite products to be placed on a firm rigorous mathematical setting - this was based on the work of Weierstrass.

Suppose that $P(x)$ is a polynomial with roots $r_1, r_2, r_3, \ldots r_n$. Then $P(x)$ can be written in the form where line (2) is obtained by multiplying out.

$$\begin{aligned} P(x) &= A(x-r_1)(x-r_2)\ldots(x-r_n) \\ &= Ax^n - A(r_1+r_2+\ldots+r_n)x^{n-1} + A\sum_{j=1}^{n}\sum_{i\neq j} r_i r_j x^{n-2} \\ &\quad + \ldots + \underbrace{(-1)^{n-1}A\sum_{j=1}^{n}\prod_{i\neq j} r_i}_{L}\, x + \underbrace{(-1)^n A\prod_{i=1}^{n} r_i}_{K}. \end{aligned}$$

Observe that if

$$K = (-1)^n A\prod_{i=1}^{n} r_i \quad \text{and} \quad L = (-1)^{n-1}A\sum_{j=1}^{n}\prod_{i\neq j} r_i$$

then

$$L = -K\sum_{i=1}^{n}\left(\frac{1}{r_i}\right)$$

and so for the special case of $K = 1$ we have

$$L = -\sum_{i=1}^{n}\left(\frac{1}{r_i}\right).$$

Observe that $K = P(0)$.

Euler thought to apply this to the sine and cosine functions, both of which have infinitely many roots. We apply it to the sine first. But since it looks like we will be dividing by the roots and 0 is a root of the sine function we'll look at $P(z) = \frac{\sin(z)}{z}$. For this function, we'll get a polynomial $\hat{P}$ that doesn't have 0 as a root:

$$\hat{P}(z) = \frac{\sin(z)}{z};$$

and since

$$\lim_{z\to 0}\frac{\sin(z)}{z} = 1$$

it is reasonable to assume that our infinite polynomial has value 1 at 0. In fact, defining $\hat{P}$ as above, where we define $\hat{P}(0) = 1$, produces a continuous polynomial-like function. Observe that in this case $K = 1$.

With this result in mind Euler considered the sine series:

$$\begin{aligned}\sin z &= z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \ldots \\ &= z(1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \frac{z^6}{7!} + \ldots)\end{aligned}$$

And thinking of the right side as a polynomial, since $\sin z = 0$ for $z = 0, \pm\pi, \pm 2\pi, \pm 3\pi, \ldots,$ these values for z are the roots of the "polynomial". Letting $w = z^2$ and dividing out the factor of z thereby removing zero from the roots (so we're now using $\hat{P}$), it follows (his reasoning indicated) that $w = \pi^2, 2^2\pi^2, 3^2\pi^2, \ldots$ are the roots of the infinite polynomial

$$\hat{P}(w) = 1 - \frac{w}{3!} + \frac{w^2}{5!} - \frac{w^3}{7!} + \ldots$$

So for this equation, $K = 1$ and the value of L, the coefficient of the w term, is $-\frac{1}{6}$ so

$$\begin{aligned}L &= -\sum_{i=1}^{n} \left(\frac{1}{r_i}\right) \\ -\frac{1}{6} &= -\left(\frac{1}{\pi^2} + \frac{1}{2^2\pi^2} + \frac{1}{3^2\pi^2} + \ldots\right) \\ \frac{\pi^2}{6} &= 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \ldots\end{aligned}$$

We already knew from Calculus II that this series converged, but we didn't know what it converged to.

Homework: Similarly one can get another result using the same "trick" with the cosine series.

I stuck in an extra page here so that you aren't tempted to look at the solution (starting on the next page) before trying it out for your self.

Solution. Although I will use the same underlying logic, I will do the details a little differently so as to get a feeling for the technique. First a repeat of some of the background about polynomials. Suppose one is given a n degree polynomial $P(x)$:

$$P(x) = A_0 + A_1 x + A_2 x^2 + \cdots + A_n x^n.$$

If $A_0 \neq 0$ we can get a new polynomial $\hat{P}(x)$ with leading term 1:

$$\begin{aligned} \hat{P}(x) = \frac{1}{A_0} P(x) &= 1 + \frac{A_1}{A_0} x + \frac{A_2}{A_0} x^2 + \ldots \frac{A_n}{A_0} x^n \\ &= 1 + B_0 x + B_2 x^2 + \ldots B_n x^n. \end{aligned}$$

Suppose now that $r_1, r_2, \ldots r_n$ are the n roots of $\hat{P}$ where roots of multiplicity $k > 1$ are repeated k times, then we claim that

$$\begin{aligned} \hat{P}(x) &= 1 + B_0 x + B_2 x^2 + \ldots B_n x^n \\ &= \Big(1 - \frac{x}{r_1}\Big)\Big(1 - \frac{x}{r_2}\Big) \ldots \Big(1 - \frac{x}{r_n}\Big). \end{aligned}$$

To justify this recall that, for two polynomials with the same roots (with the same multiplicities), that one must be a multiple of the other. Thus if they match at one point, different from the n roots, then they are the same. In our case the two expressions for the polynomial $\hat{P}$ have the same roots (this can be checked by letting $x = r_i$) and they match at $x = 0$, which is not a root so the expressions must be equal. Observe, by multiplying out, that:

$$\begin{aligned} B_1 x &= -\Big(\frac{x}{r_1} + \frac{x}{r_2} + \ldots + \frac{x}{r_n}\Big) \\ &= -x\Big(\frac{1}{r_1} + \frac{1}{r_2} + \ldots + \frac{1}{r_n}\Big). \end{aligned}$$

Now we apply these ideas to the "infinite" polynomial expression for $\cos x$ obtained from the Maclaurin expansion.

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots.$$

The roots of the cosine function are $\pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \pm\frac{5\pi}{2}, \ldots$. Then using Euler's intuitive jump, we express the cosine as an infinite product:

$$\begin{aligned} \cos x &= \Big(1 - \frac{x}{\frac{\pi}{2}}\Big)\Big(1 + \frac{x}{\frac{\pi}{2}}\Big)\Big(1 - \frac{x}{\frac{3\pi}{2}}\Big)\Big(1 + \frac{x}{\frac{3\pi}{2}}\Big) \ldots \\ &= \Big(1 - \frac{x^2}{\frac{\pi^2}{4}}\Big)\Big(1 - \frac{x^2}{\frac{3^2\pi^2}{4}}\Big) \ldots \\ &= 1 - x^2\Big(\frac{4}{\pi^2} + \frac{4}{3^2\pi^2} + \frac{4}{5^2\pi^2} + \ldots\Big) + \text{ higher powers of } x^n. \end{aligned}$$

Since the coefficient of the x^2 term in the $\cos x$ expansion is $-\frac{1}{2}$ we have:

$$\begin{aligned} -\frac{1}{2} &= -\Big(\frac{4}{\pi^2} + \frac{4}{3^2\pi^2} + \frac{4}{5^2\pi^2} + \dots\Big) \\ \frac{\pi^2}{8} &= \Big(1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \dots\Big) \\ &= \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2}. \end{aligned}$$

$\square$