**Ancient Method for Calculating Square Roots**

Suppose that $f : \mathbb{R} \to \mathbb{R}$ is a function and one is interested in finding a point of the function so that $f(p) = p$. Such a point is called a fixed point of the function. Under certain continuity conditions, if one selects a starting value $x_0$ and iterates the function to obtain successive iterates:

$$x_{n+1} = f(x_n)$$

then the sequence $\{x_n\}_{n=1}^{\infty}$ converges to the fixed point $p$. Conditions that imply this convergence include the following: that $f$ be continuously differentiable around $p$ and that $|f'(p)| < 1$. [I'll discuss this in class.]

It is conjectured (under questionable assumptions) that (circa 1,500 BCE) the Babylonians used an iterative process on the following function to calculate the square root of a number $a$:

$$f(x) = \frac{1}{2}\Big(x_n + \frac{a}{x_n}\Big).$$

Show that $p = \sqrt{a}$ is a fixed point of the function and show that $|f'(\sqrt{a})| < 1$. Around 100 CE the Greek mathematician Heron showed that this method works - it is often called Heron's method for calculating the square root of a number.

Show that this iterative process is equivalent to applying Newton's method for finding the roots of an equation to the equation $x^2 - a = 0$.

Additional exercise: come up with a function for which a similar recursive process gives you the cube root of a number.