Hyperbolic Functions.

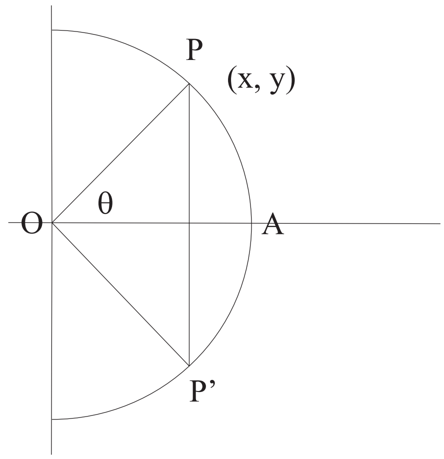

Figure 1: $x^2 + y^2 = 1$

Consider the circle $x^2 + y^2 = 1$ and let u denote the area of the sector $OPAP'$. Then the radian measure of the angle $\angle POP'$ will be 2θ. So the area u of the sector will be to the area of the circle as the length of the arc $\widehat{PAP'}$ is to the circumference of the circle:

$$\begin{aligned} \frac{u}{\pi} &= \frac{2\theta}{2\pi} \\ \theta &= u. \end{aligned}$$

So:

$$\begin{aligned} \cos u &= x \\ \sin u &= y. \end{aligned}$$

Now we consider the hyperbolic curve $x^2 - y^2 = 1$ and let u denote the area $OPAP'$. Then the area is

$$u = xy - 2\int_1^x \sqrt{t^2 - 1}\,dt$$

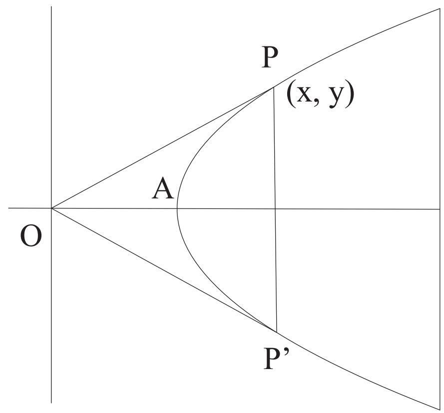

Figure 2: $x^2 - y^2 = 1$

Recall the definitions of the hyperbolic functions:

$$\begin{aligned} \sinh t &= \frac{e^t - e^{-t}}{2} \\ \cosh t &= \frac{e^t + e^{-t}}{2} \end{aligned}$$

and the following identities

$$\begin{aligned} \cosh^2 t - \sinh^2 t &= 1 \\ \sinh t &= \sqrt{\cosh^2 - 1} \end{aligned}$$

We will need the following integral

$$\begin{aligned} \int \sinh^2 t dt &= \int \Big(\frac{e^t - e^{-t}}{2}\Big)^2 \\ &= \int \frac{e^{2t} - 2 + e^{-2t}}{4} \\ &= \frac{e^{2t}}{8} - \frac{1}{2}t - \frac{e^{-2t}}{8} \\ &= \frac{e^{2t} - e^{-2t}}{8} - \frac{1}{2}t \\ &= \frac{1}{2}\Big(\frac{e^t - e^{-t}}{2}\Big)\Big(\frac{e^t + e^{-t}}{2}\Big) - \frac{1}{2}t \\ &= \frac{1}{2}\sinh t \cosh t - \frac{1}{2}t \end{aligned}$$

We now calculate the integral using the substitution $t = \cosh\theta$

$$\begin{aligned}
u &= xy - 2\int_1^x \sqrt{t^2-1}dt \\
&= xy - 2\int_0^{\cosh^{-1}x} \sqrt{\cosh^2\theta - 1}d\cosh\theta \\
&= xy - 2\int_0^{\cosh^{-1}x} \sinh\theta\sinh\theta d\theta \\
&= xy - 2\int_0^{\cosh^{-1}x} \sinh^2\theta d\theta \\
&= xy - 2\Big(\frac{1}{2}\sinh\theta\cosh\theta - \frac{1}{2}\theta\Big|_0^{\cosh^{-1}x}\Big) \\
&= xy - \Big(\sinh(\cosh^{-1}x)\cosh(\cosh^{-1}x) - \cosh^{-1}x\Big) \\
&= xy - (yx - \cosh^{-1}x) \\
&= \cosh^{-1}x \\
\cosh u &= x \quad \text{and by our identities} \\
\sinh u &= y.
\end{aligned}$$

And so the hyperbolic $\sinh u$ and $\cosh u$ are related to the area u of a "hyperbolic" sector, just as the $\sin\theta$ and $\cos\theta$ are related to the circular sector θ.