The Greeks essentially knew the geometric series:

$$1 + x + x^2 + x^3 + x^4 + \ldots = \frac{1}{1-x}$$

for values like $\frac{1}{2}$ or $\frac{1}{3}$.

Consider the series:

$$1 - 1 + 1 - 1 + 1 \ldots.$$

Mathematicians felt that it was equal to something and the natural choice was:

$$(1-1) + (1-1) + (1-1) + \ldots = 0 + 0 + 0 + \ldots = 0.$$

But it was pointed out that

$$1 - (1-1) - (1-1) - (1-1) - \ldots = 1 - 0 - 0 - 0 - \ldots = 1$$

was another natural value. Euler considered the formula

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + \ldots$$

and substituted the value -1 in the series to obtain

$$\frac{1}{2} = 1 - 1 + 1 - 1 + 1 \ldots.$$

So he felt that the value of this series was $\frac{1}{2}$.

Euler started with the series

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots$$

which, by the ratio test converges for $|x| < 1$. Then to obtain a series that converges for $|x| > 1$, he replaced the x of the above series with $\frac{1}{x}$ to obtain

$$\begin{aligned}
\ln\left(1 + \frac{1}{x}\right) &= \frac{1}{x} - \frac{1}{2x^2} + \frac{1}{3x^3} - \frac{1}{4x^4} + \ldots \\
\ln\left(\frac{1+x}{x}\right) &= \frac{1}{x} - \frac{1}{2x^2} + \frac{1}{3x^3} - \frac{1}{4x^4} + \ldots \\
\frac{1}{x} &= \ln\left(\frac{1+x}{x}\right) + \frac{1}{2x^2} - \frac{1}{3x^3} + \frac{1}{4x^4} - \ldots
\end{aligned}$$

Substitute $x = 1, 2, 3, \ldots, n$:

$$\begin{aligned}
1 &= \ln 2 + \frac{1}{2} - \frac{1}{3} + \frac{1}{4} - \ldots \\
\frac{1}{2} &= \ln\left(\frac{3}{2}\right) + \frac{1}{2\cdot 2^2} - \frac{1}{3\cdot 2^3} + \frac{1}{4\cdot 2^4} - \ldots \\
\frac{1}{3} &= \ln\left(\frac{4}{3}\right) + \frac{1}{2\cdot 3^2} - \frac{1}{3\cdot 3^3} + \frac{1}{4\cdot 3^4} - \ldots \\
\frac{1}{4} &= \ln\left(\frac{5}{4}\right) + \frac{1}{2\cdot 4^2} - \frac{1}{3\cdot 4^3} + \frac{1}{4\cdot 4^4} - \ldots \\
\vdots &= \vdots \\
\frac{1}{n} &= \ln\left(\frac{n+1}{n}\right) + \frac{1}{2\cdot n^2} - \frac{1}{3\cdot n^3} + \frac{1}{4\cdot n^4} - \ldots
\end{aligned}$$

using the property of logarithms and adding

$$\begin{aligned}
1 &= \ln 2 + \frac{1}{2} - \frac{1}{3} + \frac{1}{4} - \ldots \\
\frac{1}{2} &= \ln 3 - \ln 2 + \frac{1}{2\cdot 2^2} - \frac{1}{3\cdot 2^3} + \frac{1}{4\cdot 2^4} - \ldots \\
\frac{1}{3} &= \ln 4 - \ln 3 + \frac{1}{2\cdot 3^2} - \frac{1}{3\cdot 3^3} + \frac{1}{4\cdot 3^4} - \ldots \\
\frac{1}{4} &= \ln 5 - \ln 4 + \frac{1}{2\cdot 4^2} - \frac{1}{3\cdot 4^3} + \frac{1}{4\cdot 4^4} - \ldots \\
\vdots &= \vdots \\
\frac{1}{n} &= \ln(n+1) - \ln n + \frac{1}{2\cdot n^2} - \frac{1}{3\cdot n^3} + \frac{1}{4\cdot n^4} - \ldots
\end{aligned}$$

yields

$$\begin{aligned}
1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{n} &= \ln(n+1) + \frac{1}{2}\sum_{i=1}^{n}\frac{1}{i^2} + \frac{1}{3}\sum_{i=1}^{n}\frac{1}{i^3} + \frac{1}{4}\sum_{i=1}^{n}\frac{1}{i^4} + \ldots \\
1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{n} &= \ln(n+1) + C.
\end{aligned}$$

The constant C is called Euler's constant and in modern notation it is denoted by the Greek letter γ. It will be the following limit:

$$\begin{aligned}
\gamma &= \lim_{n\to\infty} \left(1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{n} - \ln(n+1)\right) \\
&= \lim_{n\to\infty} \left(1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{n} - \ln(n+1) + \ln n - \ln n\right) \\
&= \lim_{n\to\infty} \left(1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{n} - \ln n - \ln\left(\frac{n+1}{n}\right)\right) \\
&= \lim_{n\to\infty} \left(1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{n} - \ln n - 0\right) \\
&= \lim_{n\to\infty} \left(1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{n} - \ln n\right) \\
&\approx 0.57721566490153286060651209008240243104215933 \text{ from Wikipedia}
\end{aligned}$$

Is it not known if it is transcendental, nor even irrational. It is related to the derivative Ψ of the gamma function:

$$-\gamma = \Gamma'(1) = \Psi(1).$$