The Poincaré Model for the Hyperbolic Geometry.

Let E^2 denote the Cartesian plane. Note that we will also think of it as the complex plane for some calculations.

Let $\mathcal{D} = \{(x, y) | x^2 + y^2 < 1\}$; equivalently if we think of (x, y) as the complex number $z = x + yi$, then $\mathcal{D} = \{z|\ |z| < 1\ \}$. The points of $\mathcal{D}$ are the points in our hyperbolic geometry. So the points of this model is a subset of Cartesian or Complex plane.

Define $S^1 = \{(x, y) | x^2 + y^2 = 1\}$ or equivalently, $S^1 = \{z|\ |z| = 1\ \}$.

Definition of a line in the model: ℓ is a line if and only if 1. ℓ is the common part of $\mathcal{D}$ and a usual line in E^2 that contains the origin $(0, 0)$, or 2. ℓ is the common part of $\mathcal{D}$ and a circle perpendicular to S^1 at each intersection.

Definition. The angle between two intersecting circles is the angle between their tangent lines at the intersection point. Similarly, the angle between a line and a circle is the angle between the line and the tangent line to the circle at the point.

Definition. If w and z are two points in $\mathcal{D}$ (thought of as points in the complex plane) then the measure of the line segment $\overline{wz}$ in our model is defined as:

$$m(\overline{wz}) = \ln\left(\frac{|1 - z\overline{w}| + |z - w|}{|1 - z\overline{w}| - |z - w|}\right)$$

where on the right side of the equation z and w are interpreted as complex numbers and $\overline{w}$ denotes the complex conjugate of w.

Theorem. The Poincaré model satisfies the axioms of geometry with the hyperbolic axiom [the acute angle axiom for the Saccheri or Lambert quadrilaterals].