Tautochrone, Brachistochrone, Cycloid

Tautochrone problem. What is the shape of the curve so that the time taken by an object sliding without friction in uniform gravity to its lowest point is same no matter what starting height. (The importance is that it could be an improvement on the pendulum for a clock.)

Brachistochrone problem. What is the shape of the curve so that an object starting at rest and moving along the curve, without friction under uniform gravity, will fall to it's lowest point in the shortest time.

The (inverted) cycloid is the solution to both these problems. First a derivation of a set of parametric equations for the curve.

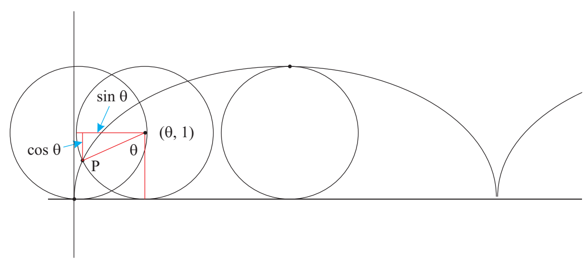

Figure 1: The cycloid.

The circle $x^2 + (y-1)^2 = 1$ is rolled with out slipping along the x-axis so that the position of the center at time θ is $(\theta, 1)$. Then the point P will have coordinates

$$P = (\theta - \sin\theta, 1 - \cos\theta).$$

In order to prove that the solution of the tautochrone problem is the cycloid, we need to derive the equation for the time of descent along a (concave up) curve. From our physics courses we know that the kinetic energy $\frac{1}{2}mv^2$ at the bottom of the curve has to equal the potential energy from the top of the curve that was used up; this potential energy is mgh. [Where m is the mass of the object, v is its terminal velocity and h is the height from the terminal level from which it was dropped.] So from the physics, where in our case h will be the y coordinate, we have

$$\begin{aligned}\frac{1}{2}mv^2 &= mgh \\ v &= \sqrt{2gh}.\end{aligned}$$

We use the notation of the 17^{th} century mathematicians. They used s for the length of the curve and the infinitesimal triangle gave them: $ds = \sqrt{dx^2 + dy^2}$; as usual the speed (velocity) is $\frac{ds}{dt}$. So

$$\begin{aligned}\frac{ds}{dt} &= v \\ ds &= vdt \\ ds &= \sqrt{2gh}dt \\ \sqrt{dx^2 + dy^2} &= \sqrt{2gh}dt\end{aligned}$$

solving for dt gives.

$$dt = \frac{\sqrt{dx^2 + dy^2}}{\sqrt{2gh}}.$$

This expression can be integrated. I'll assume that the particle starts at position $x = a$ at time t_1 and that the bottom of the curve is at $x = \pi$ at time t_2. Since we are considering the inverted cycloid (inverted about $y = 2$ the parametric form of this curve would be

$$\begin{aligned}x &= \theta - \sin\theta \\ y &= 2 - (1 - \cos\theta) = 1 + \cos\theta.\end{aligned}$$

At time 0 let θ_0 denote the value which gives the starting position of the point. The terminal position of the point is at $(\pi, 0)$ and the parameter takes on the value π at that point. As the bead slides down the curve, the value of h at some arbitrary θ will be the difference between the y value at θ_0 and the y value at θ; $h = (1 + \cos\theta_0) - (1 + \cos\theta) = \cos\theta_0 - \cos\theta$. So the time for the bead to slide down the curve will be

$$\int_{t_1}^{t_2} dt = \int_{\text{start}}^{\text{end}} \frac{\sqrt{dx^2 + dy^2}}{\sqrt{2gh}}. \tag{1}$$

Using Leibnitz notation:

$$\begin{aligned} dx &= d\theta - \cos\theta d\theta \\ dy &= -\sin\theta d\theta. \end{aligned}$$

Then substituting into the integral on the right

$$\begin{aligned} \int_{\text{start}}^{\text{end}} \frac{\sqrt{dx^2+dy^2}}{\sqrt{2gh}} &= \int_{\theta_0}^{\pi} \frac{\sqrt{(d\theta - \cos\theta d\theta)^2 + (-\sin\theta d\theta)^2}}{\sqrt{2g(\cos\theta_0 - \cos\theta)}} \\ &= \int_{\theta_0}^{\pi} \frac{\sqrt{(1-\cos\theta)^2 + (-\sin\theta)^2}}{\sqrt{2g(\cos\theta_0 - \cos\theta)}} d\theta \\ &= \int_{\theta_0}^{\pi} \frac{\sqrt{1 - 2\cos\theta + \cos^2\theta + \sin^2\theta}}{\sqrt{2g(\cos\theta_0 - \cos\theta)}} d\theta \\ &= \int_{\theta_0}^{\pi} \frac{\sqrt{2-2\cos\theta}}{\sqrt{2g(\cos\theta_0 - \cos\theta)}} d\theta \\ &= \frac{1}{\sqrt{g}} \int_{\theta_0}^{\pi} \frac{\sqrt{1-\cos\theta}}{\sqrt{\cos\theta_0 - \cos\theta}} d\theta \end{aligned} \tag{2}$$

To work toward evaluating this integral (and keeping in mind that we expect the θ_0 to disappear) we recall some helpful trig identities:

$$\begin{aligned} \cos 2\alpha = \cos^2\alpha - \sin^2\alpha \quad &\text{so} \quad \cos\theta = \cos^2\left(\frac{\theta}{2}\right) - \sin^2\left(\frac{\theta}{2}\right) \\ \cos 2\alpha = 2\cos^2\alpha - 1 \quad &\text{so} \quad \cos\theta = 2\cos^2\left(\frac{\theta}{2}\right) - 1 \\ \cos 2\alpha = 1 - 2\sin^2\alpha \quad &\text{so} \quad \cos\theta = 1 - 2\sin^2\left(\frac{\theta}{2}\right). \end{aligned}$$

Substituting in the above integral we obtain

$$\begin{aligned} \frac{1}{\sqrt{g}} \int_{\theta_0}^{\pi} \frac{\sqrt{1-\cos\theta}}{\sqrt{\cos\theta_0 - \cos\theta}} d\theta &= \frac{1}{\sqrt{g}} \int_{\theta_0}^{\pi} \frac{\sqrt{1-(1-2\sin^2(\frac{\theta}{2}))}}{\sqrt{(2\cos^2(\frac{\theta_0}{2})-1)-(2\cos^2(\frac{\theta}{2})-1)}} d\theta \\ &= \frac{1}{\sqrt{g}} \int_{\theta_0}^{\pi} \frac{\sqrt{\sin^2(\frac{\theta}{2})}}{\sqrt{\cos^2(\frac{\theta_0}{2}) - \cos^2(\frac{\theta}{2})}} d\theta \\ &= \frac{1}{\sqrt{g}} \int_{\theta_0}^{\pi} \frac{\pm\sin(\frac{\theta}{2})}{\sqrt{\cos^2(\frac{\theta_0}{2}) - \cos^2(\frac{\theta}{2})}} d\theta. \end{aligned}$$

Where the sign $\pm$ takes on the value that gives a positive integral. To solve this we use a change of variable

$$\begin{aligned}
\cos\left(\frac{\theta_0}{2}\right)u &= \cos\left(\frac{\theta}{2}\right) \\
u &= \frac{\cos(\frac{\theta}{2})}{\cos(\frac{\theta_0}{2})} \\
du &= \frac{-\frac{1}{2}\sin(\frac{\theta}{2})}{\cos(\frac{\theta_0}{2})}d\theta \\
\cos\left(\frac{\theta_0}{2}\right)du &= -\left(\frac{1}{2}\right)\sin\left(\frac{\theta}{2}\right)d\theta
\end{aligned}$$

As θ goes from θ_0 to π, u goes from 1 to 0. So performing the change of variable we obtain

$$\begin{aligned}
\frac{1}{\sqrt{g}}\int_{\theta_0}^{\pi}\frac{\pm\sin(\frac{\theta}{2})}{\sqrt{\cos^2(\frac{\theta_0}{2})-\cos^2(\frac{\theta}{2})}}d\theta &= \frac{1}{2\sqrt{g}}\int_1^0\frac{\pm\cos(\frac{\theta_0}{2})du}{\cos(\frac{\theta_0}{2})\sqrt{1-\frac{\cos^2(\frac{\theta}{2})}{\cos^2(\frac{\theta_0}{2})}}} \\
&= \frac{1}{2\sqrt{g}}\int_1^0\frac{-du}{\sqrt{1-u^2}}.
\end{aligned}$$

The negative is selected since the integral must be positive. Now, although we can solve this integral (with a sine substitution for example), there is no need to do so for the integral is independent of the starting position at θ_0. So no matter what the starting height is, the integral is the same. Thus we've proven that the cycloid is a solution to the tautochrone problem. To actually calculate the time of descent, it is easiest to evaluate the integral in equation (2) by letting our start point be the top of the cycloid: $\theta_0 = 0$ so that the integral of (2) becomes:

$$\frac{1}{\sqrt{g}}\int_{\theta_0}^{\pi}\frac{\sqrt{1-\cos\theta}}{\sqrt{\cos\theta_0-\cos\theta}}d\theta = \frac{1}{\sqrt{g}}\int_0^{\pi}\frac{\sqrt{1-\cos\theta}}{\sqrt{1-\cos\theta}}d\theta = \frac{1}{\sqrt{g}}\int_0^{\pi}1d\theta = \frac{\pi}{\sqrt{g}}.$$

Since the θ_0 disappeared in the final calculation, this tells us that the time of descent is independent of its starting position.