Test 01 Math 3010 Spring 2021
Part I Key
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Part I. Problems and exercises. Omit one problem or do all seven for extra credit.

Problem 1: Show that $r = 3i$ is a root of the following polynomial:

$$P(x) \;=\; x^4 - 2x^3 + 4x^2 - 18x - 45.$$

Then use this information to find all the roots of this polynomial.

Solution. First a proof that r is a root:

$$\begin{aligned} P(r) &= r^4 - 2r^3 + 4r^2 - 18r - 45 \\ &= (3i)^4 - 2(3i)^3 + 4(3i)^2 - 18(3i) - 45 \\ &= 81 - 2(-27i) + 4(-9) - 18(3i) - 45 \\ &= 81 + 54i - 36 - 54i - 45 \\ &= 0. \end{aligned}$$

Since the coefficients are real numbers and $3i$ is a root, then its complex conjugate $-3i$ is a root too. Since $x - r$ for a root r divides the polynomial, then $x - 3i$ and $x + 3i$ divides the polynomial. So $(x - 3i)(x + 3i) = x^2 + 9$ divides $P(x)$. Long division gives:

$$x^4 - 2x^3 + 4x^2 - 18x - 45 \;=\; (x^2 + 9)(x^2 - 2x - 5).$$

So the roots are

$$r = \pm 3i \quad \text{and} \quad r = \frac{2 \pm \sqrt{4 + 4 \cdot 5}}{2} = 1 \pm \sqrt{6}.$$

□

Problem 2: Use induction to prove the following identity about the sum of the even Fibonacci numbers:

$$\sum_{i=1}^{n} F_{2i} \;=\; F_{2n+1} - 1.$$

[Extra Credit: come up with a similar identity about the sum of the odd Fibonacci numbers and prove that your formula is correct.]

Solution. We prove the theorem by induction. Since the definition of the Fibonacci requires the addition of two previous numbers, we must test $n = 1$ and $n = 2$ for our base case.

Base case:

$$\begin{aligned}\sum_{i=1}^{1} F_{2i} &= 1 = 2 - 1 = F_3 - 1;\\ \sum_{i=1}^{2} F_{2i} &= 1 + 3 = 4 = 5 - 1 = F_5 - 1.\end{aligned}$$

We now assume that the statement is true for n and prove that this implies the statement for $n + 1$:

$$\begin{aligned}\sum_{i=1}^{n+1} F_{2i} &= \sum_{i=1}^{n} F_{2i} + F_{2(n+1)}\\ &= F_{2n+1} - 1 + F_{2(n+1)} \qquad (1)\\ &= F_{2n+1} - 1 + F_{2n+2}\\ &= F_{2n+3} - 1 = F_{2(n+1)+1} - 1. \qquad (2)\end{aligned}$$

Where step (1) follows from the induction hypothesis and step (2) follows from the definition of the Fibonacci numbers (in term of the *two* previous numbers). □

Problem 3: Use the method of Descartes to find the equation of the line tangent to the curve $f(x) = 5x^2$ at the point $(p, 5p^2)$.

Solution. The equation of the line of slope m containing the point (x_1, y_1) is:

$$\begin{aligned} y - y_1 &= m(x - x_1) \\ y &= y_1 + m(x - x_1). \end{aligned}$$

In our case $(x_1, y_1) = (p, 5p^2)$. So that gives us the equation

$$\begin{aligned} y &= 5p^2 + m(x - p) \\ &= mx + 5p^2 - mp. \end{aligned}$$

We want to know where the line intersects the curve $y = 5x^2$. To calculate the point of intersection we set the two functions equal to each other:

$$\begin{aligned} 5x^2 &= mx + 5p^2 - mp \\ 5x^2 - mx - 5p^2 + mp &= 0. \end{aligned}$$

Now this equation is supposed to have "two" roots and we know that one of them is $r = p$, so we factor out $x - p$ to obtain the other root; long division works well to obtain:

$$5x^2 - mx - 5p^2 + mp = (x - p)(5x + (5p - m)).$$

Descartes' method assumes that the root p must be a double root, a root of multiplicity 2 so that says that p must also be a root of the reduced polynomial $5x + (5p - m)$ (that it the quotient of the long division). So setting $x = p$ then the reduced polynomial is equal to 0 and we can solve for m.

$$\begin{aligned} 5p + (5p - m) &= = 0 \\ m &= 10p. \end{aligned}$$

And we know that this is the correct answer from our calculus class.

So the equation of the line will be:

$$\begin{aligned} y - y_1 &= m(x - x_1) \\ y &= 5p^2 + 10p(x - p) \\ y &= 10px - 5p^2. \end{aligned}$$

□

Problem 4: Use the infinitesimal method of Barrow, Fermat et al. to find the equation of the line tangent to the curve $f(x) = 5x^2 - 3x$ at the point $(p, f(p))$.

Solution. The technique assumes the existence of an "infinitesimal" triangle at the point $(p, f(p)) = (p, 5p^2 - 3p)$ where the triangle has base e and height $f(x+e) - f(x) = (5(p+e)^2 - 3(p+e)) - (5p^2 - 3p)$. The hypotenuse of the triangle is tangent to the curve. So the slope of the hypotenuse of this infinitesimal triangle is $f(x+e) - f(x)$ divided by e. [Barrow used the inverse of this ratio, so either way is fine, as long as you ultimately get the slope.] So the slope m will be:

$$\begin{aligned} m &= \frac{f(x+e) - f(x)}{e} \\ &= \frac{(5(p+e)^2 - 3(p+e)) - (5p^2 - 3p)}{e} \\ &= \frac{((5p^2 + 10pe + 5e^2) - 3(p+e)) - (5p^2 - 3p)}{e} \\ &= \frac{10pe + 5e^2 - 3e)}{e} \\ &= 10p + 5e - 3 \\ &= 10p - 3. \end{aligned}$$

Where the last step follows from the claim that e, and hence $5e$, is an infinitesimal. So the equation of the line with slope m containing the point $(p, 5p^2 - 3p)$ is

$$\begin{aligned} y - y_1 &= m(x - x_1) \\ y - (5p^2 - 3p) &= m(x - p) \\ y - (5p^2 - 3p) &= (10p - 3)(x - p). \end{aligned}$$

□

Problem 5: Argue that Cavalieri's principle is equivalent to Archimedes' balance method for find the volume of a solid. [You may look it up on Wikipedia.] Assume that the volume of a square based right pyramid is $\frac{1}{3}hb^2$ where h is the height and b is one side of the square base. Then use Cavalieri's principle (or Archimedes' balance method) to argue that the volume of a right circular cone with a height of h and circular base of radius r is $\frac{1}{3}\pi hr^2$.

Solution. The interpretation of Archimedes' moment arms as equal give us Cavalieri's principle. □

Problem 6: For the following polynomial, how do you "get rid of" the troublesome x^4 term? (I.e. find a transformation that transforms the equation into a quintic polynomial without the x^4 term.)

$$P(x) = x^5 + ax^4 + bx^3 + cx^2 + dx + e.$$

Solution. The technique is to let $x = t + k$ and find the k value that causes the coefficient of the t^4 term to be zero.

$$\begin{aligned} P(t+k) &= (t+k)^5 + a(t+k)^4 + \text{lower powers of } x \\ &= (t^5 + 5kt^4 + \text{lower powers of } x) \\ &\quad +(at^4 + \text{lower powers of } x) \\ &\quad +\text{lower powers of } x \\ &= t^5 + 5kt^4 + at^4 + \text{lower powers of } x \\ &= t^5 + (5k+a)t^4 + \text{lower powers of } x. \end{aligned}$$

So $k = -\frac{a}{5}$ and so the transformation that does this is:

$$x = t - \frac{a}{5}.$$

□

Problem 7: Find a root of the following polynomial.

$$P(x) = x^3 + 3x - 5.$$

Solution. From our work on the solution of the cubic we learned that the solution to the equation

$$x^3 + px = q$$

is given by

$$\begin{aligned} x &= \sqrt[3]{\frac{1}{2}\left(\sqrt{q^2 + \frac{4p^3}{27}} + q\right)} - \sqrt[3]{\frac{1}{2}\left(\sqrt{q^2 + \frac{4p^3}{27}} - q\right)} \\ \text{or } x &= \sqrt[3]{\frac{q}{2} + \left(\sqrt{\frac{q^2}{4} + \frac{p^3}{27}}\right)} + \sqrt[3]{\frac{q}{2} - \left(\sqrt{\frac{q^2}{4} + \frac{p^3}{27}}\right)}. \end{aligned}$$

Substituting $p = 3$ and $q = 5$ yields

$$x = \sqrt[3]{\frac{1}{2}\left(\sqrt{29}+5\right)} - \sqrt[3]{\frac{1}{2}\left(\sqrt{29}-5\right)}.$$

$\square$