The Definition of the Limit.

Following is the definition of limit due to Weierstrass following the work of Bolzano (1817) and Cauchy (1821)

Definition 1. *The limit of a function f.*

$$L = \lim_{x \to p} f(x)$$

means that for each positive number ϵ there is a number δ_ϵ so that:

$$\textit{If } |x - p| < \delta_\epsilon \quad \textit{then} \quad |f(x) - L| < \epsilon.$$

Equivalently: the limit of a function $f(x)$ as x approaches p is L means that for each $\epsilon > 0$ there is a $\delta_\epsilon > 0$ (which is typically dependent on ϵ) so that for all values of x within δ_ϵ of p the corresponding $f(x)$ value is within ϵ of L.

The limit game.

The Limit Point game is played on the reals $\mathbb{R}$ and starts with a function f, a point p and a number L, the function need not be defined at p. For example:

$$f(x) = \frac{\sqrt{x} - 2}{x - 4}$$

where p is the number 4 and the function is not defined at 4.

There are two players: Player $\mathcal{E}$ and player $\mathcal{D}$.

Player $\mathcal{E}$ goes first and picks a number $\epsilon > 0$. Then player $\mathcal{D}$ must pick a number $\delta > 0$; the number δ must have the property that if t is within δ of p then $f(t)$ is within ϵ of L. Algebraically:

$$\text{If } |p - t| < \delta \text{ then } |L - f(t)| < \epsilon.$$

Then the play continues turn by turn, in the n^{th} turn of play: player $\mathcal{E}$ picks a number ϵ_n then player $\mathcal{D}$ must pick a number $\delta_n > 0$ that satisfies the rules of the game ... and so on.

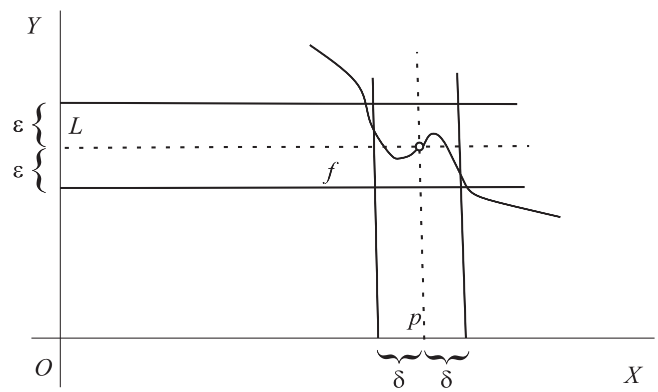

Figure 1: Player $\mathcal{D}$ selecting a good move.

Player $\mathcal{E}$ wins at the n^{th} step of the game if player $\mathcal{D}$ cannot find a number that satisfies the rules of the game. Player $\mathcal{D}$ wins if $\mathcal{D}$ can always find a $\delta_n > 0$ that satisfies the rules of the game.

It's as though $\mathcal{E}$ sets up a target for $\mathcal{D}$ to shoot at and with each turn $\mathcal{E}$ is allowed to make the target smaller and smaller.

Since this game has the possibility of being an infinite game, what is required for player $\mathcal{D}$ to win is for $\mathcal{D}$ to provide a "winning strategy." (Alternatively, one can assume that each player makes their n^{th} move in the time interval between $\frac{1}{n}$ and $\frac{1}{n+1}$ minutes before midnight and $\mathcal{D}$ wins if the game didn't end before midnight with a win on $\mathcal{E}$'s part.)

Examples. For the following sets, for each choice of p, figure out if there is a number L so that $\mathcal{D}$ will win:

$$\begin{array}{lll}
\text{a.)} & f(x) = \frac{x^2+5x-14}{x-2} & p = 2 \\
\text{b.)} & f(x) = \frac{x^3-8}{x-2} & p = 2 \\
\text{b.)} & f(x) = \frac{\sqrt{x}-2}{x-4} & p = 4 \\
\text{d.)} & f(x) = \frac{1}{x} & p = 0 \\
\text{e.)} & f(x) = x^2 & p = 2 \\
\text{f.)} & f(x) = \sqrt{x} & p = 4 \\
\text{g.)} & f(x) = \sin(\frac{1}{x}) & p = 0 \\
\text{h.)} & f(x) = x\sin(\frac{1}{x}) & p = 0.
\end{array}$$

Theorem: The function $f : (a,b) \to \mathbb{R}$ is continuous at $(p, f(p))$ if and only if $p \in (a,b)$ and player $\mathcal{D}$ has a winning strategy for the point and $L = f(p)$.

Example. Let's calculate $\frac{dy}{dx}|_3$ for the function $y = 5x^2$. First we'll use the differential technique of Leibniz to get the value of the derivative of y at $x = 3$. So we have the function $y(x) = 5x^2$.

$$\begin{aligned}
\left.\frac{dy}{dx}\right|_3 &= \left.\frac{y(x+dx)-y(x)}{dx}\right|_{x=3} \\
&= \frac{y(3+dx)-y(3)}{dx} \\
&= \frac{5(3+dx)^2 - 5\cdot 3^2}{dx} \\
&= \frac{5(9+6dx+(dx)^2)-45}{dx} \\
&= \frac{30dx+5(dx)^2}{dx} \\
&= 30+5dx \\
&= 30.
\end{aligned}$$

Where the last step follows from the fact that dx and hence $5dx$ is an infinitesimal which when added to something doesn't change its value.

Now I'll use the definition of limit due to Weierstrass to prove that this is correct. The limit definition of the derivative is

$$\lim_{\Delta x\to 0}\frac{\Delta y}{\Delta x} = \lim_{h\to 0}\frac{y(3+h)-y(3)}{h}$$

Where I use $h = \Delta x$ to make the limit look more like what we saw in Calculus I.

We want to prove that:

$$\lim_{h \to 0} \frac{y(3+h) - y(3)}{h} = 30.$$

Proof. So as the limit game's player $\mathcal{E}$ tells us, suppose $\epsilon > 0$. Then player $\mathcal{D}$ needs to pick a δ_ϵ. Player $\mathcal{D}$'s strategy is to pick $\delta_\epsilon = \frac{\epsilon}{5}$. [How they decided that will be clear when we examine the calculations.] Then when $|h - 0| < \delta$ we have:

$$\begin{aligned}
\left|\frac{y(3+h) - y(3)}{h} - L\right| &= \left|\frac{5(3+h)^2 - 5 \cdot 3^2}{h} - 30\right| \\
&= \left|\frac{5(9 + 6h + h^2) - 45}{h} - 30\right| \\
&= \left|\frac{45 + 30h + 5h^2 - 45}{h} - 30\right| \\
&= \left|\frac{30h + 5h^2}{h} - 30\right| \\
&= |30 + 5h - 30| \\
&= |5h| = 5|h| < 5\delta = 5\frac{\epsilon}{5} = \epsilon.
\end{aligned}$$

Therefore for our choice of δ, if $|h - 0| < \delta$ then

$$\left|\frac{5(3+h)^2 - 5 \cdot 3^2}{h} - 30\right| < \epsilon.$$

$\square$