Binomial Series.

Observe that $a+b$ can be put in the form $a+ax$ by setting $x=\frac{b}{a}$. Based on this fact we want to consider the binomial expansion of $(a+ax)^r$ for rational r.

First the Modern derivation of a version of the binomial theorem. Let $f(x)=(a+ax)^{\frac{m}{n}}$ then in preparation for using the Maclaurin series expansion:

$$\begin{aligned}
f(x) &= (a+ax)^{\frac{m}{n}} \\
f'(x) &= \frac{m}{n}\Big(a+ax\Big)^{\frac{m}{n}-1}a \\
f''(x) &= \frac{m}{n}\Big(\frac{m}{n}-1\Big)\Big(a+ax\Big)^{\frac{m}{n}-2}a^2 \\
f'''(x) &= \frac{m}{n}\Big(\frac{m}{n}-1\Big)\Big(\frac{m}{n}-2\Big)\Big(a+ax\Big)^{\frac{m}{n}-3}a^3 \\
&\vdots
\end{aligned}$$

from which we have

$$\begin{aligned}
f(0) &= a^{\frac{m}{n}} \\
f'(0) &= \frac{m}{n}a^{\frac{m}{n}-1}a = \frac{m}{n}a^{\frac{m}{n}} \\
f''(0) &= \frac{m}{n}\Big(\frac{m}{n}-1\Big)a^{\frac{m}{n}-2}a^2 = \Big(\frac{m(m-n)}{n^2}\Big)a^{\frac{m}{n}} \\
f'''(0) &= \frac{m}{n}\Big(\frac{m}{n}-1\Big)\Big(\frac{m}{n}-2\Big)a^{\frac{m}{n}-3}a^3 = \Big(\frac{m(m-n)(m-2n)}{n^3}\Big)a^{\frac{m}{n}} \\
&\vdots
\end{aligned}$$

From Maclaurin's theorem:

$$\begin{aligned}
f(x) &= f(0)+f'(0)x+\frac{f''(0)}{2!}x^2+\frac{f'''(0)}{3!}x^3+\ldots \\
(a+ax)^{\frac{m}{n}} &= a^{\frac{m}{n}}+\frac{m}{n}a^{\frac{m}{n}}x+\frac{m(m-n)}{2!n^2}a^{\frac{m}{n}}x^2+\frac{m(m-n)(m-2n)}{3!n^3}a^{\frac{m}{n}}x^3+\ldots
\end{aligned}$$

For a rational number $\frac{a}{b}$, Newton's formulation was:

$$(P+PQ)^{\frac{m}{n}} = \underbrace{P^{\frac{m}{n}}}_{A}+\underbrace{\frac{m}{n}AQ}_{B}+\underbrace{\frac{m-n}{2n}BQ}_{C}+\underbrace{\frac{m-2n}{3n}CQ}_{D}+\ldots$$

with

$$
\begin{aligned}
A &= P^{\frac{m}{n}} \\
B &= \frac{m}{n} AQ \\
C &= \frac{m-n}{2n} BQ \\
D &= \frac{m-2n}{3n} CQ \\
&\vdots
\end{aligned}
$$

expanding:

$$
\begin{aligned}
A &= P^{\frac{m}{n}} \\
B &= \frac{m}{n} P^{\frac{m}{n}} Q \\
C &= \frac{m(m-n)}{2n^2} P^{\frac{m}{n}} Q^2 \\
D &= \frac{m(m-n)(m-2n)}{3!n^3} P^{\frac{m}{n}} Q^3 \\
&\vdots
\end{aligned}
$$

And this matches our formulation. For the special case of

$$\frac{1}{1+x^2} = (1+x^2)^{-1}$$

replacing $P = a$ with 1 and $Q = x$ with x^2 and $m = -1$, $n = 1$ we get

$$\frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + \dots \tag{1}$$

on the other hand replacing a with x^2 and $ax = 1$ so x needs to be replaced with x^{-2} and again $m = -1$, $n = 1$ we get

$$\frac{1}{1+x^2} = x^{-2} - x^{-4} + x^{-6} - x^{-8} + \dots \tag{2}$$

Newton said to use equation (1) if $x < 1$ and use equation (2) if $x > 1$. We know from more modern consideration that series (1) converges for $|x| < 1$ and series (2) converges for $|x^{-2}| < 1$ which is equivalent to $|x| > 1$.

Another way to obtain these two series is to do polynomial long division and divide $1 + x^2$ and $x^2 + 1$ into 1 respectively.