**The Deficit.**

Definition. The deficit of a triangle $\triangle ABC$, denoted by def$(\triangle ABC)$, is defined as def$(\triangle ABC) = 180 - (\mathrm{m}(\angle CAB) + \mathrm{m}(\angle ABC) + \mathrm{m}(\angle BCA))$.

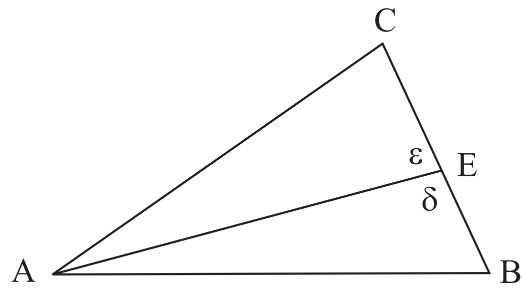


Figure 1: Deficit Theorem

We know, by theorems, def$(\triangle ABC) \geq 0$ and that there exist a triangle with zero deficit if and only if the Euclidean parallel line postulate holds.

Theorem. Given $(\triangle ABC)$ and $D \in \overline{CB}$ is between $B$ and $C$. Then:

$$\mathrm{def}(\triangle ABC) \quad = \quad \mathrm{def}(\triangle ACD) + \mathrm{def}(\triangle ADB)$$

*Proof.* The result follows from the fact that $\mathrm{m}(\angle ADC) + \mathrm{m}(\angle ADB)$ is the measure of a straight angle (i.e. 180°). □

**The AAA Theorem in the Hyperbolic Geometry.**

Theorem [Hyperbolic Geometry]. If two triangles have their angles congruent, then the triangles are congruent.

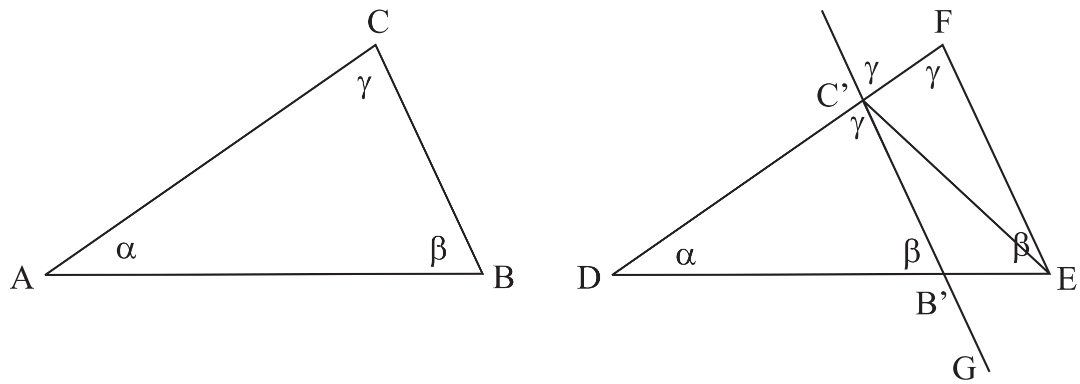


Figure 2: AAA Theorem

*Proof.* Suppose that $\triangle ABC$ and $\triangle DEF$ have corresponding angles congruent to each other, with:

$$\begin{aligned} \angle CAB &\cong \angle FDE \\ \angle ABC &\cong \angle DEF \\ \angle BCA &\cong \angle EFD. \end{aligned}$$

So the two triangles have the same deficit. Now suppose that the triangles are not congruent. Then $\overline{AC}$ is not congruent to $\overline{DF}$. We assume, without loss of generality, that m$(\overline{AC})$ < m$(\overline{DF})$. So there is a point $C'$ on $\overline{DF}$ between $D$ and $F$ so that

$$(m)(\overline{AC}) \cong (m)(\overline{DC'}.$$

Consider the line $\overleftrightarrow{C'G}$ with $G$ on the same side of $\overleftrightarrow{DF}$ as $E$ so that $\angle DC'G \cong \angle ACB$. The by Pasch's Axiom, $\overleftrightarrow{C'G}$ intersects $\overline{DE}$ or $\overline{EF}$. By the alternate interior angle theorem, $\overleftrightarrow{C'G}$ is parallel to $\overleftrightarrow{EF}$ and so must intersect $\overline{DE}$ at some point $B'$. Then, by ASA

$$\triangle ABC \cong \triangle DB'C'.$$

But then we have:

$$\begin{aligned} \mathrm{def}(\triangle DEF) &= \mathrm{def}(\triangle DB'C') + \mathrm{def}(\triangle B'C'E) + \mathrm{def}(\triangle C'EF) \\ \therefore \mathrm{def}(\triangle DEF) &> \mathrm{def}(\triangle DB'C'). \end{aligned}$$

and

$$\begin{aligned} \mathrm{def}(\triangle DEF) &= \mathrm{def}(\triangle ABC) \\ \mathrm{def}(\triangle ABC) &= \mathrm{def}(\triangle DB'C') \\ \therefore \mathrm{def}(\triangle DEF) &= \mathrm{def}(\triangle DB'C'). \end{aligned}$$

This is a contradiction.

□