Descartes' Method for finding the Tangent to a Curve.

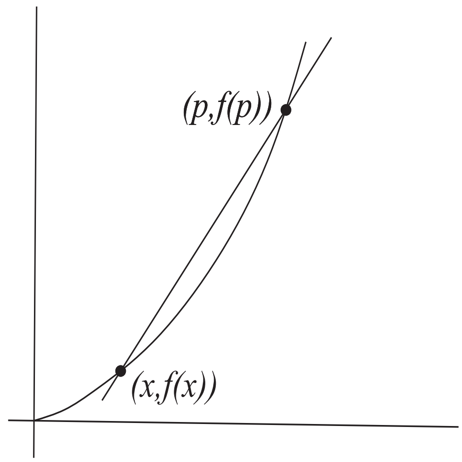

Figure 1: $y = f(x)$

René Descartes used the following method to find the slope of a line tangent to a curve. For a given curve $y = f(x)$ he considered the line of some slope m containing the point $(p, f(p))$ to which he wants to find the tangent. The equation of the line containing $(p, f(p))$ of slope m is:

$$\begin{aligned} y - y_0 &= m(x - x_0) \\ y - f(p) &= m(x - p) \\ y &= f(p) + mx - mp. \end{aligned}$$

This line intersects the curve at $(p, f(p))$ and possibly at other points. This means that p satisfies the equation $f(x) = f(p) + mx - mp$. And so p is a root of the polynomial (or more generally the function) $P(x) = f(x) - f(p) - mx + mp$. His insight is that if m could be selected so that p is a root of multiplicity at least 2, then that value of m would be the slope of the tangent line. I will show how this works with the special case of the cubic: $y = x^3$. This is the curve $f(x) = x^3$. We will find the slope of the line tangent to the curve $f(x) = x^3$ at the point (p, p^3).

We consider the polynomial $P(x) = x^3 - p^3 - mx + mp$. We know that p is a root of the polynomial so $(x - p)$ is a factor of the polynomial. Use long division to factor the polynomial and you should obtain:

$$\begin{aligned} x^3 - p^3 - mx + mp &= (x-p)(x^2+xp+p^2) - m(x-p) \\ &= (x-p)(x^2+xp+p^2-m). \end{aligned}$$

In order for p to be a double root of the polynomial $P(x)$ it must be a root of the second term of the product on the right above. So that says that, substituting p for x,

$$x^2 + xp + p^2 - m = p^2 + pp + p^2 - m = 0.$$

Solving for m gives us:

$$\begin{aligned} p^2 + pp + p^2 - m &= 0 \\ 3p^2 &= m. \end{aligned}$$

So the desired slope is $m = 3p^2$. Notice that this matches what we learned in calculus. Once you have the slope, the equation of the line tangent to the curve at the point (p, p^3) can be easily calculated.