

Euler Constant

Euler started with the series

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots$$

to obtain

$$\begin{aligned}\ln\left(1 + \frac{1}{x}\right) &= \frac{1}{x} - \frac{1}{2x^2} + \frac{1}{3x^3} - \frac{1}{4x^4} + \dots \\ \ln\left(\frac{1+x}{x}\right) &= \frac{1}{x} - \frac{1}{2x^2} + \frac{1}{3x^3} - \frac{1}{4x^4} + \dots \\ \frac{1}{x} &= \ln\left(\frac{1+x}{x}\right) + \frac{1}{2x^2} - \frac{1}{3x^3} + \frac{1}{4x^4} - \dots\end{aligned}$$

Substitute $x = 1, 2, 3, \dots, n$:

$$\begin{aligned}1 &= \ln 2 + \frac{1}{2} - \frac{1}{3} + \frac{1}{4} - \dots \\ \frac{1}{2} &= \ln\left(\frac{3}{2}\right) + \frac{1}{2 \cdot 2^2} - \frac{1}{3 \cdot 2^3} + \frac{1}{4 \cdot 2^4} - \dots \\ \frac{1}{3} &= \ln\left(\frac{4}{3}\right) + \frac{1}{2 \cdot 3^2} - \frac{1}{3 \cdot 3^3} + \frac{1}{4 \cdot 3^4} - \dots \\ \frac{1}{4} &= \ln\left(\frac{5}{4}\right) + \frac{1}{2 \cdot 4^2} - \frac{1}{3 \cdot 4^3} + \frac{1}{4 \cdot 4^4} - \dots \\ &\vdots \\ \frac{1}{n} &= \ln\left(\frac{n+1}{n}\right) + \frac{1}{2 \cdot n^2} - \frac{1}{3 \cdot n^3} + \frac{1}{4 \cdot n^4} - \dots\end{aligned}$$

using the property of logarithms and adding

$$\begin{aligned}1 &= \ln 2 + \frac{1}{2} - \frac{1}{3} + \frac{1}{4} - \dots \\ \frac{1}{2} &= \ln 3 - \ln 2 + \frac{1}{2 \cdot 2^2} - \frac{1}{3 \cdot 2^3} + \frac{1}{4 \cdot 2^4} - \dots \\ \frac{1}{3} &= \ln 4 - \ln 3 + \frac{1}{2 \cdot 3^2} - \frac{1}{3 \cdot 3^3} + \frac{1}{4 \cdot 3^4} - \dots \\ \frac{1}{4} &= \ln 5 - \ln 4 + \frac{1}{2 \cdot 4^2} - \frac{1}{3 \cdot 4^3} + \frac{1}{4 \cdot 4^4} - \dots \\ &\vdots \\ \frac{1}{n} &= \ln(n+1) - \ln n + \frac{1}{2 \cdot n^2} - \frac{1}{3 \cdot n^3} + \frac{1}{4 \cdot n^4} - \dots\end{aligned}$$

yields

$$\begin{aligned}1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{n} &= \ln(n+1) + \frac{1}{2} \sum_{i=1}^n \frac{1}{i^2} + \frac{1}{3} \sum_{i=1}^n \frac{1}{i^3} + \frac{1}{4} \sum_{i=1}^n \frac{1}{i^4} + \cdots \\1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{n} &= \ln(n+1) + C.\end{aligned}$$

The constant C is called Euler's constant and in modern notation it is denoted by γ . It will be the following limit:

$$\begin{aligned}\gamma &= \lim_{n \rightarrow \infty} \left(1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{n} - \ln(n+1) \right) \\&= \lim_{n \rightarrow \infty} \left(1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{n} - \ln(n+1) + \ln n - \ln n \right) \\&= \lim_{n \rightarrow \infty} \left(1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{n} - \ln n - \ln \left(\frac{n+1}{n} \right) \right) \\&= \lim_{n \rightarrow \infty} \left(1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{n} - \ln n - 0 \right) \\&= \lim_{n \rightarrow \infty} \left(1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{n} - \ln n \right) \\&\approx 0.5772156649015328606065120900824024310421 \text{ from Wikipedia}\end{aligned}$$

Euler first mentioned the constant in a 1734 paper. Is it not known if it is transcendental, nor even irrational. It is related to the derivative Ψ of the gamma function:

$$-\gamma = \Gamma'(1) = \Psi(1).$$