Euler Formulas

The following series representations can easily be obtained by the standard technique that produces the Maclaurin expansions of functions.

$$\begin{aligned} e^x &= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \dots \quad &(1) \\ \sin x &= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots \quad &(2) \\ \cos x &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots \quad &(3) \end{aligned}$$

Observe that if $i^2 = -1$ then:

$$\begin{aligned} i^1 &= i \\ i^2 &= -1 \\ i^3 &= -i \\ i^4 &= 1 \\ i^5 &= i \\ \vdots & \quad \vdots \end{aligned}$$

and the pattern continues. Substituting ix for x in equation (1) and also $-ix$ for x yields.

$$\begin{aligned} e^{ix} &= 1 + ix - \frac{x^2}{2!} - \frac{ix^3}{3!} + \frac{x^4}{4!} + \frac{ix^5}{5!} - \frac{x^6}{6!} - \frac{ix^7}{7!} + \dots \quad &(4) \\ e^{-ix} &= 1 - ix - \frac{x^2}{2!} + \frac{ix^3}{3!} + \frac{x^4}{4!} - \frac{ix^5}{5!} - \frac{x^6}{6!} + \frac{ix^7}{7!} + \dots . \quad &(5) \end{aligned}$$

Recognizing formulas 2 and 3 as parts of formula 4 and 5 we have:

$$\begin{aligned} e^{ix} &= \cos x + i \sin x \quad &(6) \\ e^{-ix} &= \cos x - i \sin x \quad &(7) \end{aligned}$$

Equation (6) (which implies equation (7)) is called Euler's formula - he derived it around 1740. Substituting $x = \pi$ in equation (6) we get the famous,

$$e^{i\pi} = -1.$$

This is referenced as Euler's Identity and has be called the most remarkable/beautiful identity in mathematics.

Solving (6) and (7) for sine and cosine yields:

$$\begin{aligned}\cos x &= \frac{1}{2}(e^{ix} + e^{-ix}) \\ \sin x &= \frac{1}{2i}(e^{ix} - e^{-ix}).\end{aligned}$$

Recall the definition of the hyperbolic functions and one can see a formal relationship to the trig functions:

$$\begin{aligned}\cosh x &= \frac{1}{2}(e^{x} + e^{-x}) \\ \sinh x &= \frac{1}{2}(e^{x} - e^{-x}).\end{aligned}$$

And derive the following

$$\begin{aligned}\cosh x &= \cos(ix) \\ \sinh x &= \frac{1}{i}\sin(ix) = -i\sin(ix).\end{aligned}$$

The hyperbolic trigonometric functions were introduced in the 1760's independently by Vincenzo Riccati and Johann Lambert. They have similar algebraic identities to the usual trig functions; for example:

$$\cosh^2(x) - \sinh^2 x = 1.$$

Then dividing through by cosh yields an identity relating tanh and sech.