Hyperbolic Geometry Exercises on Quadrilaterals.

For these exercises you may use the Neutral Geometry theorems. These include the congruence theorems between triangles: SAS, ASA, SSS, AAS; the alternate interior angle theorem.

Definition. The quadrilateral $\square ABCD$ is called a *Lambert quadrilateral* if it has three right angles. [Notation, for the Lambert quadrilateral $\square ABCD$ the point D is usually assumed to be the angle which has not been designated as a right angle. (Though in Euclidean geometry it will be a right angle.)]

Definition. The quadrilateral $\square ABCD$ is called a *Saccheri quadrilateral* if has two congruent sides perpendicular to a third side, called the base of the quadrilateral. [Notation, for the Saccheri quadrilateral $\square ABCD$ the side $\overline{AB}$ is usually assumed to be the base with sides $\overline{DA}$ and $\overline{CB}$ perpendicular to it.]

Saccheri Quadrilaterals.

Given: For $\square ABCD$ we have $\overline{AD} \cong \overline{BC}$, angles $\angle ABC$ and $\angle BAD$ are congruent right angles.

Exercise 1. *The summit angle of a Saccheri Quadrilateral are equal.*

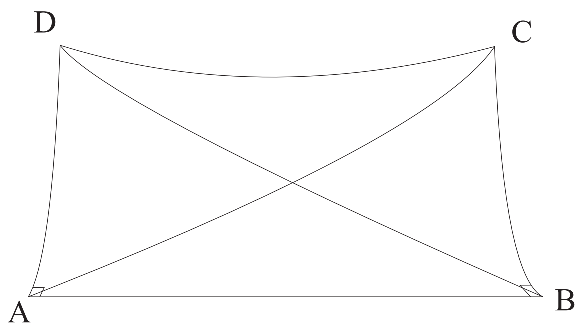

Figure 1: Saccheri Quadrilateral.

Exercise 2. *The line joining the midpoint of the base and of the summit of a Saccheri Quadrilateral is perpendicular to both of them.*

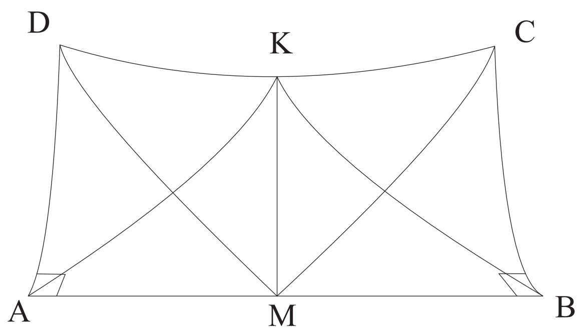

Figure 2: Saccheri Quadrilateral - Midpoint Theorem.

Exercise 3. *If perpendiculars are drawn from the extremities of the base of a triangle upon the line passing through the midpoints of the two sides, then a Saccheri quadrilateral is formed.*

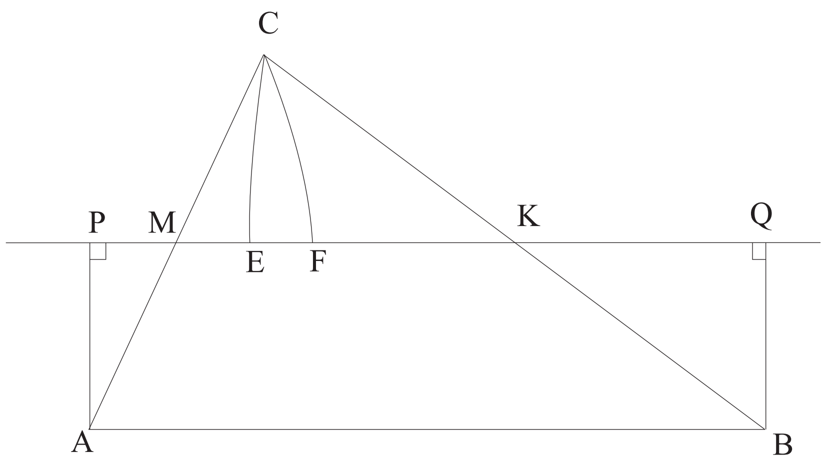

Figure 3: Triangle theorem.

Note, you are given the following:

$$\begin{aligned} \overline{AM} &\cong \overline{CM} \\ \overline{BK} &\cong \overline{CK} \\ \overline{AP} &\perp \overleftrightarrow{MK} \\ \overline{BQ} &\perp \overleftrightarrow{MK} \end{aligned}$$

Exercise 4. *The line joining the midpoints of the equal sides of a Saccheri quadrilateral is perpendicular to the line joining the midpoints of the base and summit.*

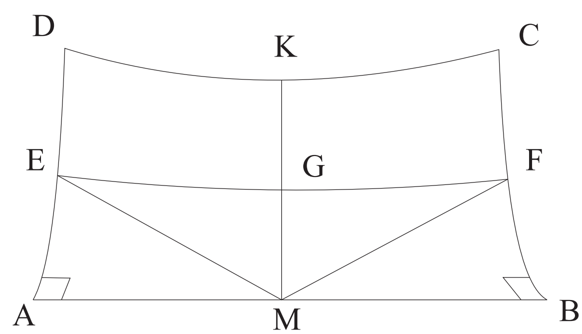

Figure 4: Midline theorem.

Exercise 5. *If* $\square ABCD$ *is a rectangle, then opposite side are congruent. [Note: this is a neutral geometry theorem.]*

Exercise 6. *If* $\square ABCD$ *is a Saccheri quadrilateral with congruent sides* $\overline{DA}$ *and* $\overline{CB}$*, then the angles* $\angle CDA$ *and* $\angle DCB$ *are congruent.*

Exercise 7. *Suppose* $\square ABCD$ *is a quadrilateral with right angles* $\angle DAB$ *and* $\angle ABC$*. Then the angle opposite the smaller side is smaller: if* $m(\overline{DA}) < m(\overline{CD})$ *then* $m(\angle ADC) > m(\angle BCD)$*.*

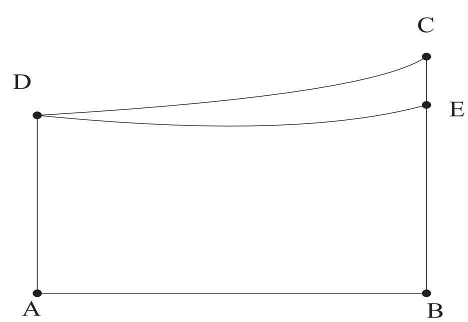

Figure 5: angle opposite the smaller side is smaller

Observe that we have the following from our exercises: Suppose $\square ABCD$ is a quadrilateral with right angles $\angle DAB$ and $\angle ABC$. Then the side opposite the larger angle is larger: if $m(\angle ADC) > m(\angle BCD)$ then $m(\overline{DA}) < m(\overline{CD})$.

Exercise 8. *Suppose ℓ and m are two parallel lines so that P and Q are points of ℓ whose distance from m are equal, then ℓ and m have a common perpendicular through the midpoint M of $\overline{PQ}$.*

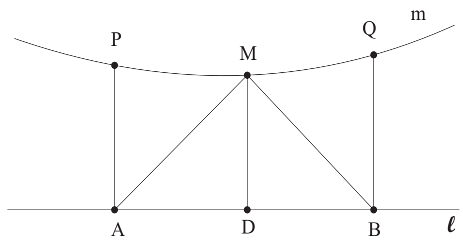

Figure 6:

Exercise 9. *On the hypothesis of the previous exercise, every other point of* ℓ *is farther from* m *than* M*.*