Hyperbolic Geometry Exercises on Quadrilaterals.

For these exercises you may use the Neutral Geometry theorems. These include the congruence theorems between triangles: SAS, ASA, SSS, AAS; the alternate interior angle theorem.

Definition. The quadrilateral $\square ABCD$ is called a *Lambert quadrilateral* if it has three right angles. [Notation, for the Lambert quadrilateral $\square ABCD$ the point D is usually assumed to be the angle which has not been designated as a right angle. (Though in Euclidean geometry it will be a right angle.)]

Definition. The quadrilateral $\square ABCD$ is called a *Saccheri quadrilateral* if has two congruent sides perpendicular to a third side, called the base of the quadrilateral. [Notation, for the Saccheri quadrilateral $\square ABCD$ the side $\overline{AB}$ is usually assumed to be the base with sides $\overline{DA}$ and $\overline{CB}$ perpendicular to it.]

Saccheri Quadrilaterals.

Given: For $\square ABCD$ we have $\overline{AD} \cong \overline{BC}$, angles $\angle ABC$ and $\angle BAD$ are congruent right angles.

Exercise 1. *The summit angle of a Saccheri Quadrilateral are equal.*

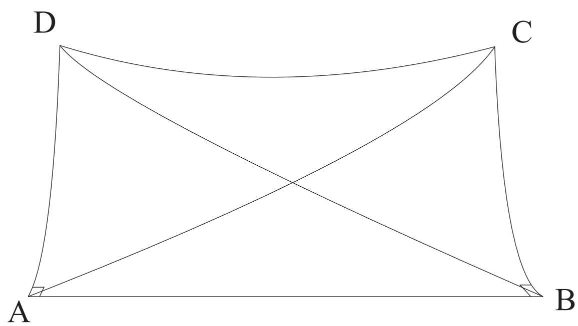

Figure 1: Saccheri Quadrilateral.

Proof. Since $\overline{AD} \cong \overline{BC}$ and $\angle ABC \cong \angle BAD$ then $\triangle ABC \cong \triangle BAD$ by SAS. So $\overline{AC} \cong \overline{BD}$ and then $\triangle DCA \cong \triangle CDB$ by SSS. This gives us $\angle CDA \cong \angle DCB$. $\square$

Exercise 2. *The line joining the midpoints of the base and of the summit of a Saccheri Quadrilateral is perpendicular to both of them.*

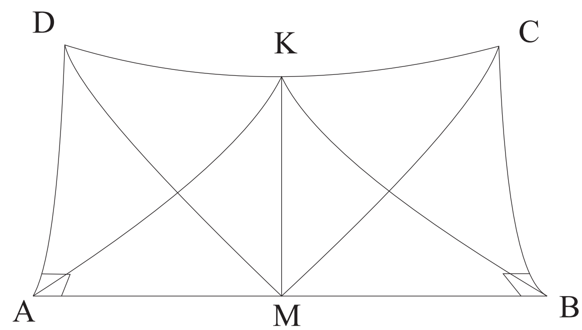

Figure 2: Saccheri Quadrilateral - Midpoint Theorem.

Proof Outline.

$$\begin{aligned}\triangle DAM &\cong \triangle CBM \quad \text{by SAS}\\ \overline{DM} &\cong \overline{CM}\\ \triangle DKM &\cong \triangle CKM \quad \text{by SSS}\\ \angle DKM &\cong \angle CKM.\end{aligned}$$

Therefore $\overline{MK}$ is perpendicular to $\overline{DC}$.

Using the similar argument (working from the top down instead of from the bottom up), we obtain the fact that $\overline{MK}$ is perpendicular to $\overline{AB}$. □

Exercise 3. *If perpendiculars are drawn from the extremities of the base of a triangle upon the line passing through the midpoints of the two sides, then a Saccheri quadrilateral is formed.*

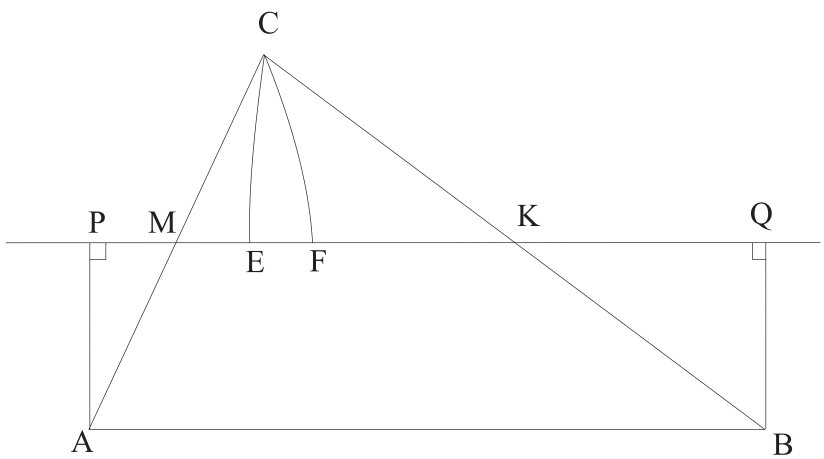

Figure 3: Triangle theorem.

Note, you are given the following:

$$\begin{aligned} \overline{AM} &\cong \overline{CM} \\ \overline{BK} &\cong \overline{CK} \\ \overline{AP} &\perp \overleftrightarrow{MK} \\ \overline{BQ} &\perp \overleftrightarrow{MK} \end{aligned}$$

Proof. Construct E so that $\overline{PM} \cong \overline{EM}$ and F so that $\overline{QK} \cong \overline{FK}$ (in the case that E and F are in the reverse order, the proof is the same). Then:

$$\begin{aligned}
\angle AMP &\cong \angle CME \text{ (vertical angles)} \\
\overline{AM} &\cong \overline{CM} \\
\overline{PM} &\cong \overline{EM} \text{ (construction)} \\
\triangle APM &\cong \triangle CEM \text{ (SAS)} \\
\angle APM &\cong \angle CEM \text{ and so is a right angle} \\
\angle BKQ &\cong \angle CKF \text{ (vertical angles)} \\
\overline{BK} &\cong \overline{CK} \\
\overline{QK} &\cong \overline{FK} \text{ (construction)} \\
\triangle BQK &\cong \triangle CFK \text{ (SAS)} \\
\angle BQK &\cong \angle CFK \text{ and so is a right angle} \\
\triangle CEF &\cong \triangle CFE \text{ (ASA)} \\
\overline{CE} &\cong \overline{CF} \\
\overline{AP} \cong \overline{CE} &\cong \overline{CF} \cong \overline{BQ}
\end{aligned}$$

So $\square PQBA$ is a Saccheri quadrilateral. $\square$

Exercise 4. *The line joining the midpoints of the equal sides of a Saccheri quadrilateral is perpendicular to the line joining the midpoints of the base and summit.*

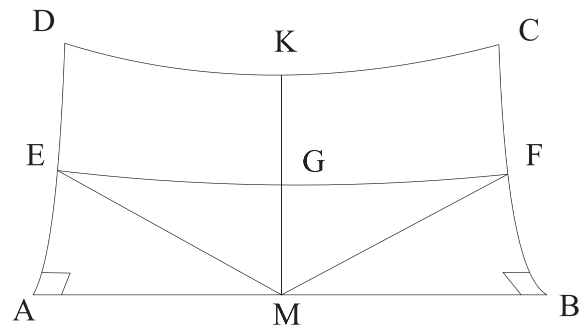

Figure 4: Midline theorem.

Exercise 5. *If $\square ABCD$ is a rectangle, then opposite side are congruent. [Note: this is a neutral geometry theorem.]*

Exercise 6. *If $\square ABCD$ is a Saccheri quadrilateral with congruent sides $\overline{DA}$ and $\overline{CB}$, then the angles $\angle CDA$ and $\angle DCB$ are congruent.*

Exercise 7. *Suppose $\square ABCD$ is a quadrilateral with right angles $\angle DAB$ and $\angle ABC$. Then the angle opposite the smaller side is smaller: if $m(\overline{DA}) < m(\overline{CD})$ then $m(\angle ADC) > m(\angle BCD)$.*

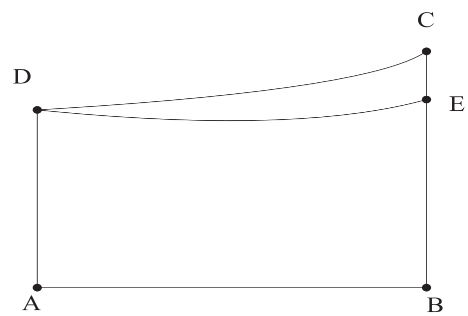

Figure 5: angle opposite the smaller side is smaller

Proof. Let E be chosen on $\overrightarrow{BC}$ so that $\overline{BE} \cong \overline{AD}$; since $m(AD) < m(BC)$ we have $B - -E - -D$.

$\angle ADE \cong \angle BED$, by Saccheri quadrilateral.

So $m(\angle ADE) < m(\angle ADC)$ since $\overrightarrow{DE}$ lies in the interior of $\angle ADC$.

$\angle BED > \angle BCD$, exterior angle theorem.

So $m(\angle ADC) > m(\angle BCD)$.

$\square$

Observe that we have the following from our exercises: Suppose $\square ABCD$ is a quadrilateral with right angles $\angle DAB$ and $\angle ABC$. Then the side opposite the larger angle is larger: if $m(\angle ADC) > m(\angle BCD)$ then $m(\overline{DA}) < m(\overline{CD})$.

Exercise 8. *Suppose ℓ and m are two parallel lines so that P and Q are points of ℓ whose distance from m are equal, then ℓ and m have a common perpendicular through the midpoint M of $\overline{PQ}$.*

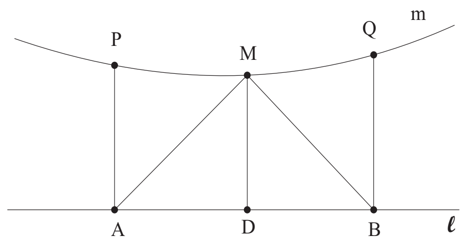

Figure 6:

Proof. Let M be the midpoint of $\overline{PQ}$ and let A, B, D be the bases of perpendicularity respectively from P, Q, M to ℓ.

$\overline{PA} \cong \overline{QB}$ by hypothesis;

$\angle APM \cong \angle BQM$, since $\square PABQ$ is a Saccheri quadrilateral;

$\overline{PM} \cong \overline{QM}$ since M is the midpoint;

so $\triangle MPA \cong \triangle MQB$ by SAS.

$\angle PMA \cong QMB$ congruencies.

$\overline{AM} \cong \overline{BM}$ congruencies;

$\overline{MB} \cong \overline{MB}$ identity;

$\angle MDA \cong MDB$ right angles;

$\triangle MDA \cong \triangle MDB$ SS -right angle.

$\angle AMD \cong \angle BMD$ congruencies;

$\angle PMD \cong \angle QMD$ angle addition of congruent angles.

Therefore $\overline{MD} \perp \overline{AB}$.

$\overline{AD} \cong \overline{BD}$ congruencies; therefore D is the midpoint of $\overline{AB}$.

□

Exercise 9. *On the hypothesis of the previous exercise, every other point of ℓ is farther from m than M.*

Proof. Referring to Figure 2: If $R \in m$ and X is the base of perpendicularity to ℓ then the quadrilateral $\square RXDM$ has right angles $\angle RXD, \angle XDM, \angle DMR$; since $def(\square RXDM > 0)$ it follows that $m(\angle MRX) < 90$. So the result follows from Claim 4. $\square$

Claim 7. Suppose that ℓ and m are lines such that there is a segment PD with $P \in m$ and $D \in \ell$ so that PD is parallel to both ℓ and m. Then m and ℓ are perpendicular and if Q and R are points of m so that $QP \cong RP$ then Q and R are the same distance from ℓ.