The Interpolation Problem; the Gamma Function

The interpolation problem: given a function with values on some discrete set, like the positive integer, then what would be a reasonable way to extend the function so that it is defined for all the reals? Euler wanted to do this for the factorial function. He presented his results in a 1729 paper. Euler concluded that

$$n! = \int_0^1 (-\ln x)^n dx.$$

Observe that (in modern calculation)

$$\begin{aligned}
\int_0^1 -\ln x \; dx &= -x \ln x + x\Big|_0^1 \\
&= 1 + \lim_{x \to 0^+} x \ln x \\
&= 1 + \lim_{x \to 0^+} \frac{\ln x}{\frac{1}{x}} \\
&= 1 + \lim_{x \to 0^+} \frac{\frac{1}{x}}{-\frac{1}{x^2}} \\
&= 1 + \lim_{x \to 0^+} \frac{-x}{1} = 1 - 0 = 1.
\end{aligned}$$

Which is consistent with $1! = 1$. Now consider the following integration,

$$\int_0^1 (-\ln x)^n dx.$$

We apply the (Leibnitz version) of the integration by parts formula $\int u \; dv = uv - \int v \; du$.

$$\begin{aligned}
dv = 1 \qquad & v = x \\
u = (-\ln x)^n \qquad & du = n(-\ln x)^{n-1}\Big(-\frac{1}{x}\Big)dx
\end{aligned}$$

$$\int_0^1 (-\ln x)^n dx = x(-\ln x)^n\Big|_0^1 + n\int_0^1 (-\ln x)^{n-1} dx$$

Since (in modern notation and using the same limit techniques as above)

$$\lim_{x \to 0^+} x(\ln x)^n = 0$$

(Euler would have probably said that $0(\ln(0))^n = 0$.) We have:

$$\int_0^1 (-\ln x)^n dx = n\int_0^1 (-\ln x)^{n-1} dx$$

which is consistent with our inductive definition of the factorial function: $n! = n \cdot (n-1)!$; so for integers the formula clearly works. Then he used the transformation $t = -\ln x$ so that $x = e^{-t}$ to obtain the following. Note that when $x = 1, t = 0$ and when $x \to 0, t \to \infty$.

$$\begin{aligned} \int_0^1 (-\ln x)^n dx &= \int_\infty^0 (t)^n d(e^{-t}) \\ &= \int_\infty^0 -(t)^n e^{-t} dt \\ &= \int_0^\infty (t)^n e^{-t} dt \end{aligned}$$

which is the modern formulation and leads us to the definition of the gamma function. For historical reasons (the fact that Gauss' π function was $\pi(n) = (n-1)!$) the gamma function Γ is defined by

$$\begin{aligned} \Gamma(n+1) &= \int_0^\infty (t)^n e^{-t} dt \\ &= n! \end{aligned}$$

Lets calculate $\Gamma(\frac{1}{2})$:

$$\begin{aligned} \Gamma(n+1) &= \int_0^\infty (t)^n e^{-t} dt \\ \Gamma\left(\frac{1}{2}\right) &= \int_0^\infty (t)^{-\frac{1}{2}} e^{-t} dt \end{aligned}$$

consider the change of variable $t = x^2$:

$$\begin{aligned}
\int_0^\infty (t)^{-\frac{1}{2}} e^{-t} dt &= \int_0^\infty (x^2)^{-\frac{1}{2}} e^{-x^2} dx^2 \\
&= \int_0^\infty \frac{1}{x} e^{-x^2} 2x dx \\
&= 2 \int_0^\infty e^{-x^2} dx \\
&= \int_{-\infty}^\infty e^{-x^2} dx \\
\Gamma\left(\frac{1}{2}\right) &= \sqrt{\pi}
\end{aligned}$$

So we need the calculation of $\int_{-\infty}^\infty e^{-x^2} dx$. I'll remind us how it's done. We will use a change of variable from rectangular to polar coordinates. So we will need $|J|$ for the Jacobian.

$$\begin{aligned}
x &= r\cos\theta \\
y &= r\sin\theta
\end{aligned}$$

$$J = \begin{bmatrix} \frac{\partial x}{\partial r} & \frac{\partial y}{\partial r} \\ \frac{\partial x}{\partial \theta} & \frac{\partial y}{\partial \theta} \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -r\sin\theta & r\cos\theta \end{bmatrix}$$

$$|J| = r\cos^2\theta + r\sin^2\theta = r$$

$$
\begin{aligned}
\left(\int_{-\infty}^{\infty} e^{-x^2} dx\right)^2 &= \left(\int_{-\infty}^{\infty} e^{-x^2} dx\right)\left(\int_{-\infty}^{\infty} e^{-y^2} dy\right) \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-x^2-y^2} dxdy \\
&= \int_{0}^{2\pi}\int_{0}^{\infty} e^{-r^2} |J| dr d\theta \\
&= \int_{0}^{2\pi}\int_{0}^{\infty} e^{-r^2} r dr d\theta \\
&= \int_{0}^{2\pi} -\frac{1}{2} e^{-r^2}\Big|_0^{\infty} d\theta \\
&= \int_{0}^{2\pi} \frac{1}{2} d\theta \\
&= \frac{1}{2}\cdot 2\pi = \pi \\
\int_{-\infty}^{\infty} e^{-x^2} dx &= \sqrt{\pi}
\end{aligned}
$$

From our probability and/or statistics class we will want to calculate

$$\int_{-\infty}^{\infty} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx$$

Using a change of variable $u = \frac{x-\mu}{\sqrt{2}\sigma}$ so that $du = \frac{1}{\sqrt{2}\sigma}$ So

$$
\begin{aligned}
\int_{-\infty}^{\infty} e^{-\frac{(x-\mu)^2}{2\sigma^2}} &= \int_{-\infty}^{\infty} e^{-u^2}\sqrt{2}\sigma du \\
&= \sqrt{2}\sigma\sqrt{\pi} = \sigma\sqrt{2\pi}
\end{aligned}
$$

Which gives us the normal distribution probability density function:

$$f_{\mu,\sigma}(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}.$$

Factorial like property of the gamma function.

From the definition of the gamma function we have the following:

$$\begin{aligned}
\Gamma(n+1) &= n! \\
\Gamma(n+2) &= (n+1)! \\
&= (n+1)\Gamma(n+1).
\end{aligned}$$

So the choice of the gamma function as a generalization of the factorial would be particular nice if we could show this for all x and not just for positive integers. We'd like to prove that $\Gamma(x) = (x-1)\Gamma(x-1)$. In the following assume x is a particular positive real number (we'll see why we need positive below). From our definition of $\Gamma(x)$ we have

$$\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}dt.$$

We proceed with integration by parts using Leibnitz notation treating x as a constant:

$$\begin{aligned}
\int udv &= uv - \int vdu \\
\Gamma(x) &= \int_0^\infty t^{x-1}e^{-t}dt \\
& \quad u = t^{x-1}; dv = e^{-t}dt \\
& \quad du = (x-1)t^{x-2}dt; v = -e^{-t} \\
\int_0^\infty t^{x-1}e^{-t}dt &= \int_0^\infty t^{x-1}e^{-t}dt \\
&= -t^{x-1}e^{-t}\Big|_0^\infty - \int_0^\infty (x-1)t^{x-2}(-e^{-t})dt.
\end{aligned}$$

It's a simple exercise to evaluate $t^{x-1}e^{-t}|_0^\infty$ as long as $x \geq 1$. When x is less than 1 the lower integration limit yields division by zero so the integration will only be finite for $x \geq 1$. In these cases we have:

$$\begin{aligned}
\int_0^\infty t^{x-1}e^{-t}dt &= -t^{x-1}e^{-t}\Big|_0^\infty - \int_0^\infty (x-1)t^{x-2}(-e^{-t})dt \\
&= 0 + \int_0^\infty (x-1)t^{x-2}e^{-t}dt \\
\Gamma(x) &= (x-1)\Gamma(x-1).
\end{aligned}$$

I add one more observation by Gauss where he generalized the gamma function to a function of a complex variable z:

$$\Gamma(z) = \frac{e^{-\gamma z}}{z} \prod_{k=1}^{\infty} \left(1 + \frac{z}{k}\right)^{-1} e^{\frac{z}{k}}.$$

The number γ is Euler's constant. Note the use of "infinite" products.