Ancient Method for Calculating Square Roots

Suppose that $f : \mathbb{R} \to \mathbb{R}$ is a function and one is interested in finding a point of the function so that $f(p) = p$. Such a point is called a fixed point of the function. Under certain continuity conditions, if one selects a starting value x_0 and iterates the function to obtain successive iterates:

$$x_{n+1} = f(x_n)$$

then the sequence $\{x_n\}_{n=1}^{\infty}$ converges to the fixed point p. Conditions that imply this convergence include the following: that f be continuously differentiable around p and that $|f'(p)| < 1$. [I'll discuss this in class.]

It is conjectured (under questionable assumptions) that (circa 1,500 BCE) the Babylonians used an iterative process on the following function to calculate the square root of a number a:

$$x_{n+1} = \frac{1}{2}\Big(x_n + \frac{a}{x_n}\Big).$$

It is thought that the reasoning went something like this: if x_n is a guess for $\sqrt{a}$ and x_n is too large, then note that $\frac{a}{x_n}$ will be too small, so the average of the two should be a better guess; similarly, if x_n is too small, then $\frac{a}{x_n}$ will be too large.

Show that $p = \sqrt{a}$ is a fixed point of the iteration function $f(x) = \frac{1}{2}(x+\frac{a}{x})$ and show that

$$|f'(\sqrt{a})| < 1.$$

Around 100 CE the Greek mathematician Heron showed that this method works - it is often called Heron's method for calculating the square root of a number.

Show that this iterative process is equivalent to applying Newton's method for finding the roots of an equation to the equation $x^2 - a = 0$.

Additional exercises.

1. Suppose a Babylonian scribe wanted to calculate $\sqrt[3]{a}$ and he uses the square root as a model to obtain:

$$x_{n+1} = \frac{1}{2}\Big(x_n + \frac{a}{x_n^2}\Big).$$

Check to make sure that the fixed point of the iterating function is the desired root; the determine if this method would work.

2. Repeat exercise 1 for the following iteration processes, in each of these cases a "weighted" average was used:

$$\begin{aligned}
\text{(a.)} \quad x_{n+1} &= \frac{1}{3}\Big(x_n + 2\frac{a}{x_n^2}\Big) \\
\text{(b.)} \quad x_{n+1} &= \frac{1}{3}\Big(2x_n + \frac{a}{x_n^2}\Big).
\end{aligned}$$

3. Of the three iteration processes, which is the most efficient and why do you think so.