Geometric interpretation of the Hyperbolic Functions.

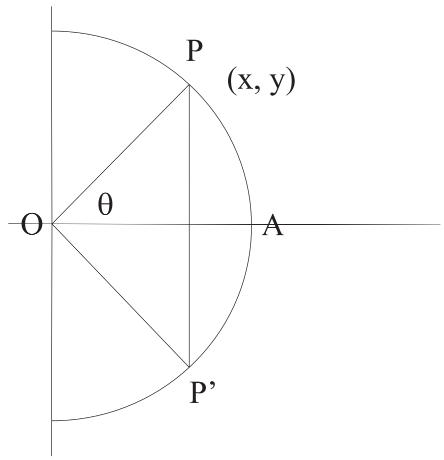

Figure 1: $x^2 + y^2 = 1$

Consider the unit circle $x^2+y^2 = 1$ and let u denote the area of the sector $OPAP'$. Then the radian measure of the angle $\angle POP'$ will be 2θ. So the area u of the sector will be to the area of the circle as the length of the arc $\widehat{PAP'}$ is to the circumference of the circle:

$$\begin{aligned} \frac{u}{\pi} &= \frac{2\theta}{2\pi} \\ \theta &= u. \end{aligned}$$

So:

$$\begin{aligned} \cos u &= x \\ \sin u &= y. \end{aligned}$$

Now we consider the "unit" hyperbolic curve $x^2 - y^2 = 1$ and let u denote the area $OPAP'$.

Then, since the area of the triangle is $\frac{1}{2}x \cdot 2y$, the area of $OPAP'$ is

$$u = xy - 2\int_1^x \sqrt{t^2 - 1}dt$$

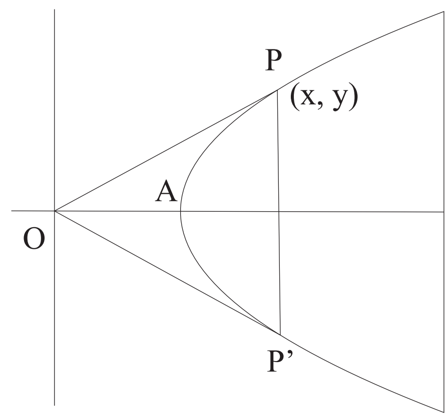

Figure 2: $x^2 - y^2 = 1$

where the integral is the area of the region below the hyperbola bounded by the x-axis and the vertical line at coordinate x.

Recall the definitions of the hyperbolic functions:

$$\begin{aligned}\sinh t &= \frac{e^t - e^{-t}}{2} \\ \cosh t &= \frac{e^t + e^{-t}}{2}\end{aligned}$$

and the following identities

$$\begin{aligned}\cosh^2 t - \sinh^2 t &= 1 \\ \sinh t &= \sqrt{\cosh^2 - 1}\end{aligned}$$

We will need the following integral

$$\begin{aligned}
\int \sinh^2 t dt &= \int \left(\frac{e^t - e^{-t}}{2}\right)^2 dt \\
&= \int \frac{e^{2t} - 2 + e^{-2t}}{4} dt \\
&= \frac{e^{2t}}{8} - \frac{1}{2}t - \frac{e^{-2t}}{8} \\
&= \frac{e^{2t} - e^{-2t}}{8} - \frac{1}{2}t \\
&= \frac{1}{2}\left(\frac{e^t - e^{-t}}{2}\right)\left(\frac{e^t + e^{-t}}{2}\right) - \frac{1}{2}t \\
&= \frac{1}{2}\sinh t \cosh t - \frac{1}{2}t
\end{aligned}$$

We now calculate the integral using the substitution $t = \cosh\theta$

$$\begin{aligned}
u &= xy - 2\int_1^x \sqrt{t^2 - 1} dt \\
&= xy - 2\int_0^{\cosh^{-1} x} \sqrt{\cosh^2\theta - 1} d\cosh\theta \\
&= xy - 2\int_0^{\cosh^{-1} x} \sinh\theta \sinh\theta d\theta \\
&= xy - 2\int_0^{\cosh^{-1} x} \sinh^2\theta d\theta \\
&= xy - 2\left(\frac{1}{2}\sinh\theta\cosh\theta - \frac{1}{2}\theta\Big|_0^{\cosh^{-1} x}\right) \\
&= xy - \Big(\sinh(\cosh^{-1} x)\cosh(\cosh^{-1} x) - \cosh^{-1} x\Big) \\
&= xy - (yx - \cosh^{-1} x) \\
&= \cosh^{-1} x.
\end{aligned}$$

This gives us $\cosh u = x$ and by our identities $\sinh u = y$.