Infinite Series

Zeno of Elea (490 - 430 BCE) considered infinite series, though he, as did the Greeks at the time, assumed that the sum of an infinite series (of positive terms - they did not have negative numbers) is infinite, or unsummable. The dichotomy paradox is the following (I quote Wikipedia): "*Suppose Atalanta wishes to walk to the end of a path. Before she can get there, she must get halfway there. Before she can get halfway there, she must get a quarter of the way there. Before traveling a quarter, she must travel one-eighth; before an eighth, one-sixteenth; and so on.*" The paradox is that the infinite sum

$$\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\dots$$

should add up to 1 but, according to their reasoning, it doesn't.

The Greeks essentially knew the geometric series:

$$1+x+x^2+x^3+x^4+\dots=\frac{1}{1-x}$$

for values like $\frac{1}{2}$ or $\frac{1}{3}$. Archimedes used a version of it in his "Quadrature of the Parabola."

Euclid had a version of the following in his "Elements":

$$a+ar+ar^2+ar^3+\dots+ar^n \quad = \quad \frac{a(1-r^{n+1})}{1-r}.$$

Gregory (1638-1675) discovered the series for the trigonometric functions. (He also had a proof for a special version of the Fundamental Theorem of Calculus.) In 1687 (July 5) Newton published his *Philosophiæ Naturalis Principia Mathematica*, in Latin, of course, where he made use of the binomial theorem; he had generalized it in 1665 for non-integer values as an infinite series.

Colin Maclaurin and Brooks Taylor, whose work we study in Calculus II, did their work in the early 18^{th} century.

There was some interesting activity in India regarding infinite series. Mādhava of Sangamagrāma (1340 - 1425) founded the Kerala School of Astronomy and Mathematics. He studied infinite series and obtained the sine and cosine series. He used the arc-tangent series to obtain the following (now) well know series for π:

$$\frac{\pi}{4} \quad = \quad 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots.$$

The school ended around 1632.

Because there was no reasonable definition for the sum of an infinite series in the 1700's, mathematicians didn't agree on certain infinite sums. Consider the series:

$$1 - 1 + 1 - 1 + 1 \ldots$$

18^{th} century mathematicians felt that it was equal to something and the natural choice was:

$$(1 - 1) + (1 - 1) + (1 - 1) + \ldots = 0 + 0 + 0 + \ldots = 0.$$

But it was pointed out that

$$1 - (1 - 1) - (1 - 1) - (1 - 1) - \ldots = 1 - 0 - 0 - 0 - \ldots = 1$$

was another natural value. Euler considered the formula

$$\frac{1}{1 - x} = 1 + x + x^2 + x^3 + x^4 + \ldots$$

and substituted the value -1 in the series to obtain

$$\frac{1}{2} = 1 - 1 + 1 - 1 + 1 \ldots$$

So he felt that the value of this series was $\frac{1}{2}$.